

Spinon Deconfinement in Quantum Critical 2 + 1 D Antiferromagnets

Zaira Nazario^{*} and David I. Santiago[†]

^{*} Max Planck Institute for the Physics of Complex systems, Nöthnitzer Str. 38, 01187 Dresden, Germany

[†] Instituut-Lorentz for Theoretical Physics, Universiteit Leiden,
P. O. Box 9506, NL-2300 RA Leiden, The Netherlands

(Dated: February 6, 2008)

The Néel magnetization of 2+1 D antiferromagnets is composed of quark-like spin 1/2 constituents, the spinons, as follows from the CP^1 mapping. These quark spinons are confined in both the Néel ordered phase and quantum paramagnetic phases. The confinement in the quantum paramagnetic phase is understood as arising from quantum tunneling events, instantons or hedgehog monopole events. In the present article, we study the approach to the quantum critical point, where the quantum paramagnetic phase ceases to exist. We find that irrespective of the intrinsic spin of the antiferromagnet, instanton events disappear at the deconfined critical point because instanton tunnelling becomes infinitely costly and have zero probability at the quantum critical point. Berry phase terms relevant to the paramagnetic phase vanish at the quantum critical point, but make the confinement length scale diverge more strongly for half-integer spins, next strongest for odd integer spins, and weakest for even integer spins. There is an emergent photon at the deconfined critical point, but the “semimetallic” nature of critical spinons screens such photon making it irrelevant to long distance physics and the deconfined spinons are strictly free particles. A unique prediction of having critical free spinons is an anomalous exponent η for the susceptibility exactly equal to one. Experimentally measurable response functions are calculated from the deconfined spinon criticality.

PACS numbers: 75.10.-b, 75.40.Cx, 75.40.Gb, 75.40.-s

I. INTRODUCTION

Shortly after the dawning days of renormalization group studies¹ of thermodynamic critical phenomena (continuous finite temperature phase transitions), this work was generalized to quantum critical phenomena (continuous zero temperature phase transitions²) induced by tuning parameters of the underlying Hamiltonian rather than the temperature. Since the mid 1970s², quantum phase transitions have attracted ever increasing theoretical and experimental activity. Quantum critical behavior has been obtained from quantum fluctuations of the order parameter^{2,3}. It is then concluded that systems with d spatial dimensions have quantum critical points identical to thermal critical points in $d + z$ dimensions when the time direction scales as z space dimensions. In this traditional approach, the quantum transition is studied via the Wilson renormalization group in which fluctuations of the order parameter are taken properly into account. This is the Landau-Ginzburg-Wilson (LGW) approach. On the other hand, some measurements can be interpreted as casting doubt on such a picture⁴. In particular, critical exponents are coming out different than what is predicted. *The exponents are not those of the classical $d + z$ theory with order parameter fluctuations only.*

There have been recent suggestions^{5,6} that there will be quantum critical physics which do not follow from LGW order parameter fluctuations alone^{5,6}. The new physics consists of the existence low energy elementary excitations intrinsic to, and existing only at the critical point, which will contribute and can modify the quantum critical properties. It was postulated that these excitations will be fractionalized^{5,6}. That quantum critical points

will have unique eigenstates is generally true as long as the critical propagator has an anomalous exponent. Such excitations are expected to be fractionalized, but they need not be so in all cases. These critical degrees of freedom provide critical fluctuations beyond those of the order parameter fluctuations which are usually included in the standard Ginzburg-Landau-Wilson (LGW) phase transition lore.

There are aspects of continuous phase transitions universal to both classical thermodynamic criticality and quantum mechanical criticality. Both types of transitions are characterized by a diverging length scale as it is impossible for a macroscopic system to qualitatively change behavior unless there are arbitrarily large scale fluctuations or correlations, either thermal, quantum or both^{1,2,7}. This diverging length scale makes the critical properties universal and independent of microscopic details, except for the most general details like symmetry and dimensionality. The diverging correlations make the system respond to external stimuli in a scale invariant manner.

The scale invariance universal to both thermal and quantum transitions is characterized by critical exponents. To be somewhat more explicit, we concentrate in relativistic quantum critical points, but we emphasize that this physics can take place in other systems. For such a system, which we take to be an antiferromagnet, we are interested in the Néel magnetization Green's function, or staggered magnetic susceptibility. We will think of a transition between a Néel ordered (antiferromagnetic) and disordered (paramagnetic) phase.

In the ordered phase the transverse Green's function or susceptibility corresponds to spin wave propagation and it has a nonanalyticity in the form of a pole corresponding

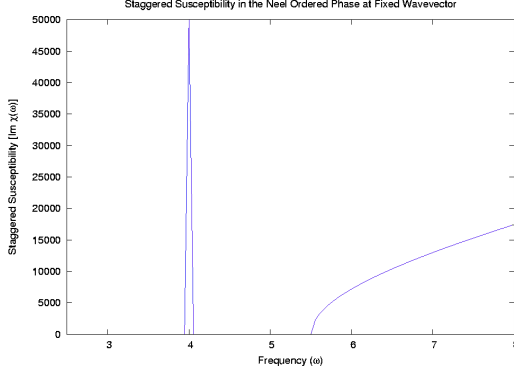


FIG. 1: Density of states in the Néel ordered phase.

to such propagation:

$$\langle \vec{n}(-\omega, -\vec{k}) \cdot \vec{n}(\omega, \vec{k}) \rangle = \frac{Z(\omega, \vec{k})}{c^2 k^2 - \omega^2} + G_{\text{incoh}}(\omega, \vec{k}). \quad (1)$$

Here $Z(\omega, \vec{k})$ is between 0 and 1, and the incoherent background G_{incoh} vanishes at long wavelengths and small frequencies. The pole structure of the Green's function is clearly illustrated when one plots the imaginary part of the Néel magnetization propagator as shown in figure 1. The fact that the Green's function has a pole means that transverse Goldstone spin waves are low energy eigenstates of the antiferromagnet. At criticality, the system has no Néel order and thus Goldstones cannot be elementary excitations of the system.

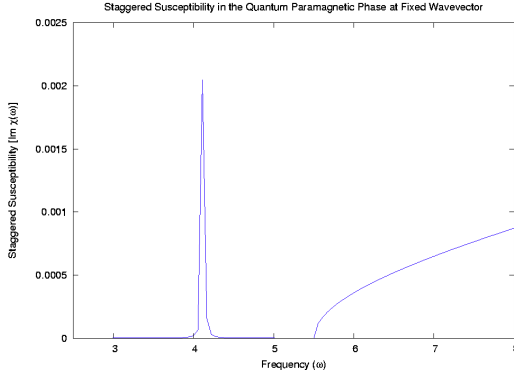


FIG. 2: Density of states in the quantum paramagnetic phase.

In the disordered phase the Green function or susceptibility corresponds to spin wave propagation with all three polarizations and it has a pole nonanalyticity corresponding to such propagation:

$$\langle \vec{n}(-\omega, -\vec{k}) \cdot \vec{n}(\omega, \vec{k}) \rangle = \frac{A(\omega, \vec{k})}{c^2 k^2 + \Delta^2 - \omega^2} + G_{\text{incoh}}(\omega, \vec{k}). \quad (2)$$

Here $A(\omega, \vec{k})$ is between 0 and 1, and the incoherent background G_{incoh} vanishes at long wavelengths and small

frequencies, Δ is the gap to excitations in the disordered phase. The pole structure of the Green's function is clearly illustrated when one plots the imaginary part of the Néel magnetization propagator as shown in figure 2. That this Green's function has a pole means that triplet or triplon spin waves are low energy eigenstates of the disordered antiferromagnet. For 2+1 D antiferromagnets, and in general for antiferromagnets below the upper critical dimension, the quasiparticle pole residue A vanishes as the system is tuned to the quantum critical point^{8,9}. At criticality, triplon excitations have no spectral weight and thus triplons cannot be elementary excitations of the system.

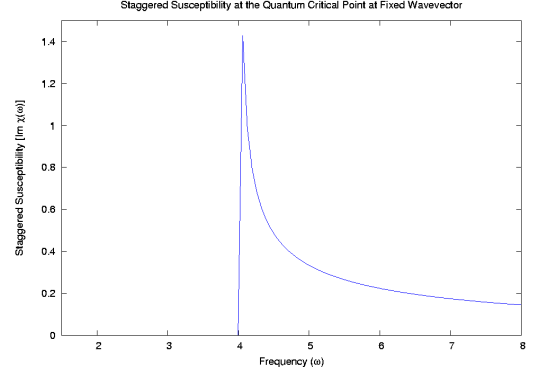


FIG. 3: Density of states at the quantum critical point.

On the other hand right at criticality the response function below the upper critical dimension (below which $\eta \neq 0$, while above $\eta = 0$) has nonanalyticities that are worse than poles

$$\langle \vec{n}(-\omega, -\vec{k}) \cdot \vec{n}(\omega, \vec{k}) \rangle = A' \left(\frac{1}{c^2 k^2 - \omega^2} \right)^{1-\eta/2} \quad (3)$$

as obtained from the renormalization group studies of the nonlinear sigma model^{8,10,11,12}. Below the upper critical dimension η is a nonintegral universal number for each dimensionality. This critical susceptibility has no pole structure, but has a branch cut. It sharply diverges at $\omega = ck$ and has an imaginary part for $\omega > ck$. The nonanalytic structure of the Green's function is clearly illustrated when one plots the imaginary part of the Néel magnetization propagator as shown in figure 3. Branch cuts in quantum many-body or field theory represent immediate decay of the quantity whose Green function is being evaluated. Hence the elementary excitations or eigenstates of the noncritical quantum mechanical phases break up as soon as they are produced when the system is tuned to criticality: they do not have integrity. The complete lack of pole structure and the branch cut singularity below the upper critical dimension mean that the elementary excitations of the quantum mechanical phases away from criticality, the spin waves, *cannot even be approximate eigenstates* at criticality as they are absolutely unstable.

The quantum critical point is a unique quantum mechanical phase of matter, which under any small perturbation becomes one of the phases it separates. It is a repulsive fixed point of the renormalization group. As far as the transition from one quantum mechanical phase to the other is continuous, and both phases have different physical properties, the critical point will have its unique physical properties different from the phases it separates. The properties of the critical point follow from the critical Hamiltonian $H(g_c)$ (g_c is the critical coupling constant), which will have a unique ground state and a collection of low energy eigenstates which are its elementary excitations. These low energy eigenstates are different from those of each of the phases as long as we are below the upper critical dimension. *As a matter of principle, all quantum critical points below the upper critical dimension will have their intrinsic elementary excitations.*

We have seen that below the upper critical dimension, the excitations of the stable quantum phases of the system become absolutely unstable and decay when the system is tuned to criticality. The question comes to mind immediately: what could they be decaying into? When one tries to create an elementary excitation of one of the phases, it will decay immediately into the elementary excitations of the critical point. The critical excitations will be bound states of the excitations of the stable phases the critical point separates. These bound states could be fractionalized as conjectured by Laughlin⁵ and Senthil, *et. al.*⁶, but they need not be in all cases. These critical degrees of freedom are responsible for corrections to the LGW phase transition canon⁶. *The intrinsic quantum critical excitations contribute to the thermodynamical and/or physical properties of the quantum critical system.*

In the present work, we develop these ideas to find if they occur in antiferromagnets. We will consider the disordered paramagnetic phase of $2 + 1$ D antiferromagnets and the approach to the quantum critical point from such a phase. These are described by the nonlinear sigma model augmented by Berry phase terms as originally discovered by Haldane^{13,14} and developed by others^{15,16,17}:

$$\mathcal{Z} = \int \mathcal{D}\vec{n} \delta(\vec{n}^2 - 1) e^{-S/\hbar} \quad (4)$$

with

$$S = S_B + \int_0^\beta \frac{d(c\tau)}{2ga} \int d^2\vec{r} \left[(\nabla_{\vec{r}}\vec{n})^2 + \frac{1}{c^2} (\partial_\tau\vec{n})^2 \right]. \quad (5)$$

$g = 2\sqrt{2}/S$ is the dimensionless coupling constant, a is the lattice constant, S is the microscopic spin with \hbar included (not to be confused with the Euclidean action), and S_B is the Berry phase term. Their effect is nonzero only in the disordered phase of the Heisenberg antiferromagnet as first suggested by Haldane¹⁴ and worked out by Read and Sachdev¹⁸. In particular, when the microscopic spins in the lattice are half-odd integers, the paramagnetic ground state has a spin Peierls bond order that

breaks the lattice symmetry and is four-fold degenerate. For odd integer spins the paramagnetic ground state has a spin Peierls bond order that breaks the lattice symmetry and is two-fold degenerate. For even integer spins we have a valence bond solid that does not break the lattice symmetry. All of these quantum paramagnetic phases have a spin triplet gapped excitation, the triplon, and a spin zero gapped collective mode.

Motivated by apparent anomalies and interesting effects in the physics of cuprate superconductors, whose parent state is a Mott insulating $2 + 1$ D antiferromagnet, Laughlin suggested⁵ that the Néel field would break up into constituents “quark” spinons at the quantum critical point between a Néel ordered and a quantum paramagnetic phase. A couple of years ago, Fisher, Sachdev and collaborators⁶ suggested that such a physics indeed occurs, but only in spin $1/2$, $2 + 1$ D antiferromagnets.

The quantum paramagnetic phase of antiferromagnets is equivalent to a “charged” CP^1 spinon field whose fictitious $U(1)$ charge couples to an emergent $U(1)$ gauge field or photon generated by the fluctuations of the CP^1 field^{19,20}. The CP^1 mapping ($\vec{n} = z^\dagger \vec{\sigma} z$, with $\vec{\sigma}$ the vector of sigma matrices) is obtained from the nonlinear sigma model description of antiferromagnets in terms of the Néel field \vec{n} ¹³. Haldane discovered that in $1 + 1$ and $2 + 1$ D, the nonlinear sigma model needs to be augmented by Berry phase terms in the paramagnetic or spin disordered phase^{13,14,18}, which in $2 + 1$ D leads to breaking of lattice symmetries for odd integers and half odd integer spins^{14,18}. If one doubts the nonlinear sigma model mapping of antiferromagnets in the disordered phase, Read and Sachdev¹⁸, starting from the Heisenberg model, showed that the disordered phase is indeed described by “charged” CP^1 “quark” spinons z coupled to an emergent photon A_μ .

In $2 + 1$ D, the CP^1 model has important tunneling events²¹ which correspond to instanton hedgehog events that effectively make the $U(1)$ gauge field compact. Moreover, if we start from the appropriate Heisenberg lattice description, the gauge group is necessarily compact. Polyakov showed^{22,23} that compact QED confines as the Wilson loop²⁴ obeys an area law when instanton or monopole events are included. Polyakov’s proof did not include matter, but it is believed¹⁸, and we show below, that the presence of charged matter does not eliminate the tunneling instanton events *as long as the charged matter is massive*. Therefore, spinons are confined in the paramagnetic phase and are thus closely analogous to the quarks of Quantum Chromodynamics.

Fisher, Sachdev and collaborators suggested that the Berry phase-induced quadrupling of instanton events intrinsic to spin $1/2$ antiferromagnets makes such monopole events irrelevant at the critical point between the paramagnetic and Néel ordered phases⁶. Therefore, spinons are deconfined at such a critical point for spin $1/2$ antiferromagnets. They further suggested that at the deconfined quantum critical point the critical exponent η of the Néel field correlator is due to the decay of the Néel field into

the deconfined quark spinons. These spinons will be the intrinsic excitations of the quantum critical point.

We study here the question of deconfinement, its consequences and how it occurs. We find that when deconfinement of the Néel field into two spin 1/2 quarks occurs, *the critical exponent η is exactly equal to 1 regardless of the origin of deconfinement*. Deconfinement will occur whenever instanton events vanish. We find that for all values of the microscopic spin, integers and half-odd integers, instanton events vanish at the quantum critical point where the paramagnetic phase ceases to exist. The vanishing of instantons happens for two reasons. The masslessness of the spinons at criticality screens the instanton fields, making them irrelevant at long distances. Furthermore, *for all values of the microscopic spins*, the Euclidean action of instanton events becomes infinite at the quantum critical point due to the masslessness of the spinons²¹, and thus *the probability of instanton events at criticality is zero. Hence deconfinement occurs independent of the value of the microscopic spin*.

On the other hand, there is some dependence on the microscopic spin on the quantum paramagnetic phase as the lattice symmetry breaking depends on the spin value through Berry phase terms¹⁸. We also found further dependence on the microscopic spin as the system is tuned to the quantum critical point since our analysis also has the consequence that the confinement length will diverge faster upon approach to criticality for half-odd integer spins, next fastest for odd integer spins, and slowest for even integer spins as a consequence of the Berry phase terms relevant to the quantum paramagnetic phase.

For the first time, we write down the effective critical theory and from it, calculate experimental consequences. At criticality, there is still a $U(1)$ -mediated gauge interaction between massless spinons, so in principle they might not be free. We find that at criticality, spinons are very mobile because of their masslessness. Hence they screen very effectively their gauge interaction so they are strictly free at long distances. The decay of the Néel field into two free particles leads to a critical exponent of $\eta = 1$ exactly. This result can be used as a diagnostic of deconfined criticality, that is, *for a free deconfined spinon critical point we predict a critical exponent η exactly equal to one*.

The deconfined critical points studied and elucidated here seem to be different than the 2 + 1 D Heisenberg critical points. It has been suggested before that these two different types of critical points might occur in 2 + 1 D⁶. One particular suggestion is that interactions irrelevant to the Néel and quantum paramagnetic phases turn the Heisenberg critical point into a deconfined critical point and there seems to be indirect numerical evidence for such physics²⁵. Before making strong conclusions one must wait for experimental evidence and/or further and more explicit numerical evidence.

II. CP^1 MAPPING OF THE $O(3)$ NONLINEAR SIGMA MODEL

We consider the CP^1 mapping of $SO(3)$ vectors \vec{n} ^{19,20}

$$\vec{n} = z^\dagger \vec{\sigma} z \quad (6)$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad (7)$$

with the restriction

$$|z|^2 \equiv |z_1|^2 + |z_2|^2 = 1 \quad (8)$$

inherited from $\vec{n}^2 = 1$. The z 's are bosonic. Notice that the CP^1 map has 4 variables. The restriction $|z|^2 = 1$ eliminates one, leaving 3 independent variables. This seems to be one too much as the $O(3)$ nonlinear sigma model has only 2 independent variables since one is eliminated by the nonlinear condition. This is not so as one of the variables is redundant because of the gauge symmetry $z \rightarrow ze^{i\theta}$. The z 's are spinors, i.e. spin 1/2 objects. The z 's are the quark-like spinon constituents of the Néel field \vec{n} .

With this mapping the $O(3)$ nonlinear sigma model partition function becomes

$$\mathcal{Z} = \int \mathcal{D}z \mathcal{D}z^\dagger \delta(|z|^2 - 1) e^{-S} \quad (9)$$

with Euclidean action

$$S = \frac{2}{ga} \int d^3r (|\partial_\mu z|^2 - |z^\dagger \partial_\mu z|^2) + S_B. \quad (10)$$

and Berry phase

$$S_B = \sum_i \epsilon_i \int_0^\beta d\tau z^\dagger \partial_\tau z. \quad (11)$$

The lattice regularization is necessary to define the Berry phase, as the Berry phase has microscopic lattice sensitivity. We can decouple the quartic term via a Hubbard-Stratonovich transformation leading to^{19,20}

$$\mathcal{Z} = \int \mathcal{D}z \mathcal{D}z^\dagger \mathcal{D}A_\mu \delta(|z|^2 - 1) e^{-S} \quad (12)$$

$$S = \frac{2}{ga} \int d^3r |(\partial_\mu - iA_\mu)z|^2 + S_B \quad (13)$$

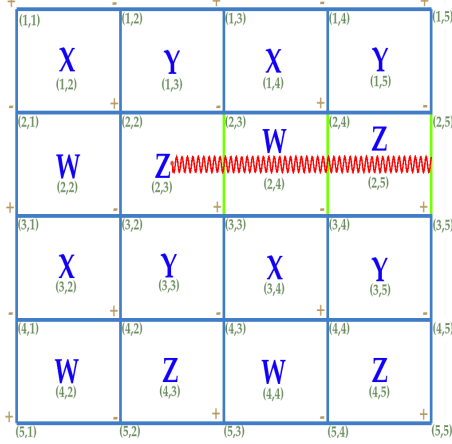
Now the gauge invariance is explicit as the kinetic term for the CP^1 fields is built up of covariant derivatives. Since the action is quadratic in A_μ , the saddle point evaluation about the minimum with respect to A_μ is exact and can be used to go back to the original action by substituting the solution to the A_μ equation of motion

$$A_\mu = \frac{i}{2} [z \partial_\mu z^\dagger - z^\dagger \partial_\mu z] = iz \partial_\mu z^\dagger = -iz^\dagger \partial_\mu z \quad (14)$$

In $2 + 1$ dimensions, which is our main concern here, the Berry phase terms can be written in terms of the gauge fields as¹⁸

$$S_B = \sum_s i\pi S \zeta_s q_s \quad (15)$$

where S is the microscopic spin (not to be confused with the Euclidean action), s are points in the dual lattice around which a hedgehog or magnetic monopole of strength q_s is centered and ζ_s is 0, 1, 2 and 3 depending on which dual lattice, W, X, Y, Z , the monopole is centered.



q_s is given by

$$q_s = \frac{1}{2\pi} \int_{\Sigma} dS_{\mu\nu} F_{\mu\nu} \quad (16)$$

where the normal to the surface Σ is closed and $F_{\mu\nu}$ is the Maxwell tensor for A_{μ} . Such spinon induced gauge field tunneling event cannot happen unless the gauge field is compact. Some readers might object that the nonlinear sigma model from which we started should not apply in the paramagnetic phase. For those readers we point out that one can start from the Heisenberg model and use a Schwinger boson representation on the lattice and one obtains a similar action, and leads to the same CP^1 model with the advantage that in such a derivation the emergent gauge field is necessarily compact. In fact, such a derivation was performed by Read and Sachdev¹⁸.

We can write the delta function in the partition function in its integral form, so that

$$\mathcal{Z} = \int \mathcal{D}z^{\dagger} \mathcal{D}z \mathcal{D}A_{\mu} \mathcal{D}\lambda e^{-S} \quad (17)$$

where now

$$S = \frac{2}{ga} \int d^3r \left\{ |(\partial_{\mu} - iA_{\mu})z|^2 + i\lambda \frac{ga}{2} (z^{\dagger} z - 1) \right\} + S_B. \quad (18)$$

Integrating the z 's we obtain an effective action for the A 's and the λ 's. This effective action has a minimum at

$A = 0$ and $\lambda = \text{constant}$. Varying the effective action with $A = 0$ and $\lambda = \text{constant}$ with respect to λ gives a self consistency equation for λ

$$1 - \frac{4\pi^2}{g} = \frac{m}{\Lambda} \arctan\left(\frac{\Lambda^2}{m^2}\right) \quad (19)$$

where $\Lambda = 1/a$ is the inverse lattice constant and $\langle i\lambda \rangle = 2m^2/(ga)$. Since λ is a mass term for the z 's, the nonlinear condition $|z|^2 = 1$ generates a mass m for the spinons z fields for $g \geq g_c \equiv 4\pi^2$. g_c corresponds to the quantum critical point where the quantum paramagnetic phase dies and in all probability gives rise to the Néel ordered phase.

The A_{μ} seems to play a passive role, as it has no dynamics of its own in the bare action. This is not so once fluctuations and dynamical corrections are calculated. The gauge field acquires dynamics through the z field fluctuations leading to an effective action with a Maxwell action for the gauge fields. In fact the long wavelength gauge field propagator gives²⁰

$$\langle A_{\mu}(-k_{\alpha}) A_{\nu}(k_{\alpha}) \rangle_{1\text{-loop}} = \frac{2}{3\pi m} (k_{\mu} k_{\nu} - \delta_{\mu\nu} k^2). \quad (20)$$

This term is the same as would be obtained from a term in the action of the form

$$\frac{1}{3\pi m} \int d^3r F_{\mu\nu}^2 \equiv \frac{1}{4e^2} \int d^3r F_{\mu\nu}^2. \quad (21)$$

Hence $e^2 \simeq 3m\pi/4$ in two dimensions. Higher order corrections will renormalize e^2 such that $e^2 = m h(g, m/\Lambda)$, where h is dimensionless. Higher order corrections also generate terms that vanish at long wavelengths and are thus dropped. Therefore, the effective action is one for massive spinon fields z coupled to a $U(1)$ gauge field

$$S = \frac{2}{ga} \int d^3r \left\{ |(\partial_{\mu} - iA_{\mu})z|^2 + m^2 |z|^2 + i\delta\lambda \frac{ga}{2} (|z|^2 - 1) \right\} + \int \frac{d^3r}{4e^2} F_{\mu\nu}^2 + S_B. \quad (22)$$

We then have a theory of “charged” spinons coupled to an emergent compact $U(1)$ gauge field or photon^{18,19,20}.

III. EFFECTIVE ACTION FOR THE QUANTUM PARAMAGNETIC PHASE OF $2 + 1$ D ANTIFERROMAGNETS

For the paramagnetic or disordered phase of $2 + 1$ D antiferromagnets, $g > g_c$, we have seen that the spinon fields z acquire a gap. The quantum critical point corresponds to the spinons becoming massless at $g = g_c$. We also saw that the adiabatic fluctuations of the spinons generate dynamics for the gauge fields. We have an emergent compact $U(1)$ gauge field coupled to a complex spinor representing the spinon fields. In our case the gauge field is compact as follows from the lattice

Schwinger boson derivation of the effective CP^1 theory for the Heisenberg model. Therefore, the quantum paramagnetic phase and quantum critical point of 2 + 1 D antiferromagnets is mapped to compact 2 + 1 D QED with two complex matter fields with an $SU(2)$ internal symmetry. The internal symmetry is effectively the original invariance of the nonlinear sigma model and plays a passive role for the most interesting properties of the paramagnetic phase to be developed in this section.

If the emergent photon did not correspond to a compact $U(1)$ group, the paramagnetic phase would have massive spinon excitations and massless photons. On the other hand, on careful thought, this seems completely wrong as the paramagnetic phase is fully gapped and the emergent photon would be gapless. It turns out that compact QED in 2+1 D, as originally shown by Polyakov^{22,23}, has monopole tunneling events, usually called instantons or hedgehogs, such that the ground state of the theory is a monopole condensate. Such a ground state gives the photon a mass *without breaking the gauge symmetry* and makes the Wilson loop obey an area rather than a perimeter law²⁴, rendering the theory confining. Polyakov's proof did not include matter. We show below that including the charged matter spinon fields, the theory still confines and the spectrum is fully massive and singlet with respect to the $U(1)$ gauge field.

Before moving to study the effects of instantons, we remind the reader that we do not have pure QED, but we have the gauge fields coupled to charge spinon fields z , which satisfy the nonlinear condition $|z|^2 = 1$ as enforced by the Lagrange multiplier $\delta\lambda$. In order to find the effect of the spinons on the theory, they can be integrated out of the partition function explicitly to yield

$$Z = \int \mathcal{D}(\delta\lambda) \mathcal{D}A_\mu e^{-S} \times \det^{-2} \left[-\frac{2}{ga} (\partial_\mu - iA_\mu)^2 + i\delta\lambda + \frac{2}{ga} m^2 \right] \quad (23)$$

with

$$S = \int d^3r \left[\frac{1}{4e^2} F_{\mu\nu}^2 - i\delta\lambda \right] + S_B. \quad (24)$$

Using $\det M = \exp[\text{tr} \ln M]$, the partition function can be rewritten as

$$Z = \int \mathcal{D}(\delta\lambda) \mathcal{D}A_\mu e^{-S_{eff}} \quad (25)$$

with

$$S_{eff} = S_B + \int d^3r \left[\frac{1}{4e^2} F_{\mu\nu}^2 - i\delta\lambda \right] + 2\text{tr} \ln \left[-\frac{2}{ga} (\partial_\mu - iA_\mu)^2 + i\delta\lambda + \frac{2}{ga} m^2 \right]. \quad (26)$$

For the moment we only consider the $\delta\lambda$ dependent part of the action and its path integration. We can expand about the saddle point $\delta\lambda = 0$ and path integrate

over $\delta\lambda$ to obtain

$$\det M \int \mathcal{D}(\delta\lambda) \exp[-\delta\lambda M \delta\lambda] = \det M^{1/2} \quad (27)$$

where the operator M is the inverse of the square of the operator $-2(\partial_\mu - iA_\mu)^2/(ga) + 2m^2/(ga)$. Hence the total partition function after taking care of the constraint, i.e. after integrating over $\delta\lambda$, is

$$Z = \int \mathcal{D}A_\mu e^{-S_{f1}} \\ S_{f1} = S_B + \int \frac{d^3r}{4e^2} F_{\mu\nu}^2 + \text{tr} \ln \left[-\frac{2}{ga} (\partial_\mu - iA_\mu)^2 + \frac{2}{ga} m^2 \right]. \quad (28)$$

This is equivalent to

$$Z = \int \mathcal{D}A_\mu \det^{-1} \left[-\frac{2}{ga} (\partial_\mu - iA_\mu)^2 + \frac{2}{ga} m^2 \right] e^{-S_{f2}} \\ S_{f2} = \int \frac{d^3r}{4e^2} F_{\mu\nu}^2 + S_B \quad (29)$$

or disentangling the determinant in terms of a complex scalar field Φ

$$Z = \int \mathcal{D}A_\mu e^{-S_{f3}} \\ S_{f3} = \frac{2}{ga} \int d^3r \left[|(\partial_\mu - iA_\mu)\Phi|^2 + m^2 \Phi^* \Phi \right] + \frac{1}{4e^2} \int d^3r F_{\mu\nu}^2 + S_B. \quad (30)$$

Hence antiferromagnets in 2 + 1 D, which were shown to be equivalent to compact QED coupled to spinor spinon fields z with the constraint $|z|^2 = 1$ become equivalent to 2 + 1 D compact QED coupled to charged complex scalar fields once the constraint is enforced.

Now the complex scalar fields can be integrated out to yield their effect on the theory. When they are integrated out their main effect is to screen electromagnetic fields by a factor ϵ_m which at long distances takes the form

$$\epsilon_m = 1 + \frac{e^2}{m} f \left(\frac{m}{\Lambda}, \frac{e^2}{\Lambda} \right) \quad (31)$$

where $f > 0$. The largest contributions to the screening effects of the Φ , and hence the spinons, at long distances can be easily calculated by computing and summing polarization diagrams. To one loop order, in the Landau gauge, we obtain for the long wavelength photon propagator

$$\langle A_\mu(-k) A_\nu(k) \rangle \simeq \frac{1}{k^2} \left(1 - \frac{e^2}{3\pi m} \right) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (32)$$

Hence we see that the spinons decrease the effective magnetic charges and thus have a diamagnetic screening effect which we parametrize by a screening constant

ϵ_m . The most important diagrams contributing to this screening are the bubble diagrams, which we sum to obtain

$$\begin{aligned} \langle A_\mu(-k)A_\nu(k) \rangle &= \frac{1}{\epsilon_m k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \\ &\simeq \frac{1}{[1 + e^2/(3\pi m)] k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \end{aligned} \quad (33)$$

Therefore in the random phase approximation (RPA), $\epsilon_m = 1 + e^2/(3\pi m)$. We finally obtain that the effective action including the effects of spinons is given by screening electromagnetic fields by the appropriate factor ϵ_m

$$S = \frac{1}{4e^2\epsilon_m} \int d^3r F_{\mu\nu}^2 + S_B. \quad (34)$$

IV. CONFINEMENT OF SPINONS IN THE QUANTUM PARAMAGNETIC PHASE OF 2 + 1 D ANTIFERROMAGNETS

Now that we have seen that once the effects of the spinons is included, the effective action for the antiferromagnet is equivalent to compact QED with an appropriate screening factor induced by the spinons, we need to take into account the nontrivial tunneling effects arising from the compactness of the gauge group^{18,22,23}. The monopole tunneling events considered by Polyakov create electromagnetic fields according to

$$\nabla \times \vec{H} = 0, \quad \nabla \cdot \vec{H} = 2\pi q \delta(\vec{r} - \vec{r}') \quad (35)$$

where

$$H_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda} F_{\nu\lambda} \quad (36)$$

and q is necessarily an integer as follows from the compactness of the gauge field. These tunneling events correspond to certain tunneling events for the spinons via the mapping

$$A_\mu = \frac{i}{2} [z \partial_\mu z^\dagger - z^\dagger \partial_\mu z] = i z \partial_\mu z^\dagger = -i z^\dagger \partial_\mu z \quad (37)$$

In fact, such spinon tunneling events have been calculated by Murthy and Sachdev²¹ and found to have Euclidean action given by

$$S_{cl} = 2\rho(q) \ln(\Lambda/m) \quad (38)$$

leading to the instanton fugacity

$$\zeta = e^{-S_{cl}} = \exp[-2\rho(q) \ln(\Lambda/m)] = \left[\frac{m}{\Lambda} \right]^{2\rho(q)}. \quad (39)$$

This instanton action depends on the charge q of the tunneling event. For single instanton event $\rho \equiv \rho(1) \simeq 0.06229609$.

From the form above we see that the instanton events would have a Coulomb law field

$$H_\mu = \sum_a \frac{q_a [r_\mu - r_\mu^a]}{2|r - r^a|^3} \quad (40)$$

where q_a are the charges of the tunneling events centered at r_μ^a in 2 + 1 D space-time. If A_{cl}^μ are the classical gauge fields corresponding to such instanton configurations, a semiclassical expansion around such configurations with gauge fields $A_{cl}^\mu + A^\mu$ yields the effective Maxwell action, Berry phases and instanton Coulomb gas action

$$\begin{aligned} &\left[\int d^3r \frac{1}{4e^2\epsilon_m} F_{\mu\nu}^2 \right] + i \sum_s \pi S \zeta_s q_s \\ &+ \frac{\pi}{2e^2\epsilon_m} \sum_{s \neq t} \frac{q_s q_t}{[(\vec{R}_s - \vec{R}_t)^2 + (\tilde{\tau}_s - \tilde{\tau}_t)^2]^{1/2}} \end{aligned} \quad (41)$$

where $F_{\mu\nu}$ is the Maxwell field of A_μ . In order to get the partition function of the antiferromagnet, we have to integrate over the Euclidean times of the instanton events and sum over the instanton events centered in the dual lattice. The partition function for our system is given by

$$\mathcal{Z} = \int \mathcal{D}z \mathcal{D}z^\dagger \mathcal{D}A_\mu e^{-S_{zA}} \mathcal{Z}_{inst} \quad (42)$$

where the Euclidean non-instanton action is

$$S_{zA} = \frac{1}{4e^2\epsilon_m} \int d^3r F_{\mu\nu}^2 \quad (43)$$

and

$$\mathcal{Z}_{inst} = \sum_{K, q_s} \frac{\zeta^K}{K!} \prod_{s=1}^K \left[\sum_{R_s} \int_0^\beta \frac{d\tau_s}{a} \right] e^{-S_{inst}} \quad (44)$$

with Euclidean instanton action

$$\begin{aligned} S_{inst} &= \frac{\pi}{2e^2\epsilon_m} \sum_{s \neq t} \frac{q_s q_t}{[(\vec{R}_s - \vec{R}_t)^2 + (\tilde{\tau}_s - \tilde{\tau}_t)^2]^{1/2}} \\ &+ i \sum_s \pi S \zeta_s q_s. \end{aligned} \quad (45)$$

Note that this is a Coulomb gas action augmented by Berry phases.

A. Mapping of the Instanton Coulomb Gas to Sine-Gordon Theory Including the Effects of Berry Phases

As we have just seen, the instanton contributions will provide constant average field fluctuations in the ground state, and this is described by a Coulomb gas partition function. This Coulomb gas can be disentangled by using the fact that the Laplacian operator is the inverse of the Coulomb potential. Following closely the methods of Polyakov, and Read and Sachdev, which included the effects of Berry phases for the first time^{18,22,23}, we obtain

$$\begin{aligned}
\mathcal{Z}_{inst} &= \int \mathcal{D}\chi \left[\exp \left\{ -\frac{e^2 \epsilon_m}{8\pi} \int_0^\beta d\tau \left[\sum_{\langle s,t \rangle} (\chi_s - \chi_t)^2 + \sum_s a^2 \left[\frac{\partial \chi_s}{\partial \tau} \right]^2 \right] \right\} \sum_K \frac{\zeta^K}{K!} \prod_{s=1}^K \int_0^\beta \frac{d\tau_s}{a} \sum_{R_s, q_s} \exp \left[i \sum_{q_s} [\pi S \zeta_s + \chi_s] q_s \right] \right] \\
&= \int \mathcal{D}\chi \left[\exp \left\{ -\frac{e^2 \epsilon_m}{8\pi^2} \int_0^\beta d\tau \left[\sum_{\langle s,t \rangle} (\chi_s - \chi_t)^2 + \sum_s a^2 \left[\frac{\partial \chi_s}{\partial \tau} \right]^2 \right] \right\} \sum_K \frac{\zeta^K}{K!} \left(\int_0^\beta \frac{d\tau_s}{a} \sum_{R_s} 2 \cos [\pi S \zeta_s + \chi_s] \right)^K \right] \\
&= \int \mathcal{D}\chi \exp \left\{ -\frac{e^2 \epsilon_m}{8\pi^2} \int_0^\beta d\tau \left[\sum_{\langle s,t \rangle} (\chi_s - \chi_t)^2 + \sum_s a^2 \left[\frac{\partial \chi_s}{\partial \tau} \right]^2 \right] + 2\zeta \int_0^\beta \frac{d\tau_s}{a} \sum_{R_s} \cos [\pi S \zeta_s + \chi_s] \right\} \quad (46) \\
&= \int \mathcal{D}\chi \exp \left\{ -\frac{e^2 \epsilon_m a^2}{8\pi^2} \int_0^\beta d\tau \left[\sum_{\langle s,t \rangle} \left(\frac{\chi_s - \chi_t}{a} \right)^2 + \sum_s \left[\frac{\partial \chi_s}{\partial \tau} \right]^2 - \frac{16\pi^2 (ma)^{2\rho}}{e^2 \epsilon_m a^3} \sum_s \cos [\pi S \zeta_s + \chi_s] \right] \right\}
\end{aligned}$$

where $\rho = 0.06229609$ and we sum over instantons with charges $q = 0, 1, -1$ since higher monopole charges are strongly suppressed as they have very small fugacity or probability^{21,22,23}. Hence, as found by Polyakov, we see that the Coulomb gas is equivalent to a Sine-Gordon theory. The only difference is that the Sine-Gordon theory we obtain, which is centered in the dual lattice, is frustrated by the Berry phase terms since the argument of the cosine has the phase shift $\pi S \zeta_s$ with the values $0, \pi S, 2\pi S$ and $3\pi S$ depending on whether the instanton is in the W, X, Y or Z sublattice^{14,18}.

We first study the case of even integer spin. In such a case the phase shift $\pi S \zeta_s$ in the cosine of the Sine-Gordon

theory obtained from instanton events is a multiple of 2π and thus equivalent to zero.

We now consider the case of odd integer S . In that case the phase shift of the cosine of the Sine-Gordon theory is 0 for χ_s in the W dual sublattice, and an even integer for χ_s in the Y dual sublattice. Therefore, the W and Y lattice are equivalent and we shall call it Y . Similarly, for χ_s in the X or Z sublattice the cosine in the Sine-Gordon have phase shifts of odd multiples of π and thus are equivalent. We shall call them X . The Sine-Gordon action then becomes

$$S_{sg} = 2 \frac{e^2 \epsilon_m a^2}{8\pi^2} \int_0^\beta d\tau \left\{ \sum_{\langle s,t \rangle} \left(\frac{\chi_s^X - \chi_t^Y}{a} \right)^2 + \sum_s \left[\left(\frac{\partial \chi_s^X}{\partial \tau} \right)^2 + M^2 \cos \chi_s^X \right] + \sum_t \left[\left(\frac{\partial \chi_t^Y}{\partial \tau} \right)^2 - M^2 \cos \chi_t^Y \right] \right\}. \quad (47)$$

where $M^2 = [16\pi^2 (ma)^{2\rho}] / [e^2 \epsilon_m a^3]$ Defining

$$\chi^X = \chi_1 + \chi_2, \quad \chi^Y = \chi_1 - \chi_2 \quad (48)$$

and going to the continuum limit, the Sine-Gordon action can be written as

$$S_{sg} = 2 \frac{e^2 \epsilon_m}{8\pi^2} \int_0^\beta \int d^2 \vec{r} \left[(\nabla \chi_1)^2 + 4\Lambda^2 \chi_2^2 - 2\sqrt{2} \Lambda \chi_1 (\nabla \chi_2)^2 + \left(\frac{\partial \chi_1}{\partial \tau} \right)^2 + \left(\frac{\partial \chi_2}{\partial \tau} \right)^2 - M^2 \sin \chi_1 \sin \chi_2 \right]. \quad (49)$$

Since χ_2 has a mass of the order of the cutoff $\Lambda = 1/a$, all gradients and time derivatives of χ_2 are suppressed to zero. We must still minimize the action with respect to χ_2 . The minimization gives

$$\chi_2 = \frac{M^2}{8\Lambda^2} \sin \chi_1 \cos \chi_2 \simeq \frac{M^2}{8\Lambda^2} \sin \chi_1 + \mathcal{O} \left(\frac{M^6}{\Lambda^6} \right) \quad (50)$$

which, when substituted into the Sine-Gordon action gives the effective low energy action

$$S_{sg} = 2 \frac{e^2 \epsilon_m}{8\pi^2} \int d^3 \vec{r} \left[(\partial_\mu \chi_1)^2 - \frac{M^4}{32\Lambda^2} \cos(2\chi_1) \right] \quad (51)$$

after the phase shift $\chi_1 \rightarrow \chi_1 + \pi/2$. This is also a Sine-Gordon model and leads to the same long distance instanton physics and confinement as found for even integers. On the other hand, there is a microscopic difference as this model with odd spins breaks the lattice symmetry, leading to a two-fold degenerate ground state. Both models, besides the usual triplon excitations once confinement is introduced, will also have a massive spin zero mode, χ in the first model and χ_1 in the latter.

We now move to the case of odd half integral spin. In this case the lattice symmetry is broken and the ground

state is four-fold degenerate. The cosine term in the original Sine-Gordon theory has a phase shift of zero for ζ_s and χ_s in the dual W sublattice, a phase shift of $\pi/2$ for the fields in the dual X sublattice, π for the fields in the dual Y sublattice, and $3\pi/2$ for the fields in the dual Z sublattice. Actually these phase shifts are for a spin 1/2. For spin 3/2 the phase shifts are interchanged among the four sublattices, but lead to similar physics by relabeling of the sublattices (basically interchanging X and Z). Exactly similar four-fold degeneracies happen for all half-odd integer spins. The action in this case is

$$\begin{aligned}
S_{sg} = & \frac{e^2 \epsilon_m a^2}{8\pi^2} \int_0^\beta \frac{d\tau}{a^2} \sum_{\langle s,t \rangle} \left\{ (\chi_s^X - \chi_t^Y)^2 + (\chi_s^X - \chi_t^W)^2 + (\chi_s^Y - \chi_t^Z)^2 + (\chi_s^W - \chi_t^Z)^2 \right\} \\
& + \frac{e^2 \epsilon_m a^2}{8\pi^2} \int_0^\beta d\tau \sum_t \left[\left(\frac{\partial \chi_t^W}{\partial \tau} \right)^2 + \left(\frac{\partial \chi_t^X}{\partial \tau} \right)^2 - M^2 \cos \chi_t^W + M^2 \sin \chi_t^X \right] \\
& + \frac{e^2 \epsilon_m a^2}{8\pi^2} \int_0^\beta d\tau \sum_t \left[\left(\frac{\partial \chi_t^Y}{\partial \tau} \right)^2 + \left(\frac{\partial \chi_t^Z}{\partial \tau} \right)^2 + M^2 \cos \chi_t^Y - M^2 \sin \chi_t^Z \right]
\end{aligned} \tag{52}$$

Defining

$$\begin{aligned}
\chi_W &= \chi_1 + \chi_2 + \chi_3, & \chi_X &= \chi_1 - \chi_2 + \chi_4 \\
\chi_Y &= \chi_1 + \chi_2 - \chi_3, & \chi_Z &= \chi_1 - \chi_2 - \chi_4
\end{aligned} \tag{53}$$

we obtain

$$\begin{aligned}
S_{sg} = & 4 \frac{e^2 \epsilon_m}{8\pi^2} \int d^3 r \left\{ (\nabla \chi_1)^2 + 8\Lambda^2 \chi_2^2 + 2\Lambda^2 \chi_3^2 + 2\Lambda^2 \chi_4^2 - 2\Lambda \chi_1 (\nabla \chi_4) \right. \\
& \left. + \left(\frac{\partial \chi_1}{\partial \tau} \right)^2 + \left(\frac{\partial \chi_2}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial \chi_3}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial \chi_4}{\partial \tau} \right)^2 + \frac{1}{2} M^2 [\sin(\chi_1 + \chi_2) \sin \chi_3 + \cos(\chi_1 - \chi_2) \sin \chi_4] \right\}.
\end{aligned} \tag{54}$$

Since χ_2, χ_3 and χ_4 have masses of the order of the cutoff Λ , all gradients and time derivatives of χ_2, χ_3 and χ_4 are suppressed to zero. We now minimize the action with respect to χ_3 and χ_4 to get

$$\begin{aligned}
\chi_3 &= -\frac{M^2}{8\Lambda^2} \sin(\chi_1 + \chi_2) \cos \chi_3 \\
&\simeq -\frac{M^2}{8\Lambda^2} \sin(\chi_1 + \chi_2) + \mathcal{O}\left(\frac{M^6}{\Lambda^6}\right) \\
\chi_4 &= -\frac{M^2}{8\Lambda^2} \cos(\chi_1 - \chi_2) \cos \chi_4 \\
&\simeq -\frac{M^2}{8\Lambda^2} \cos(\chi_1 - \chi_2) + \mathcal{O}\left(\frac{M^6}{\Lambda^6}\right).
\end{aligned} \tag{55}$$

Substitution of these expressions in the action yields

$$S_{sg} = 4 \frac{e^2 \epsilon_m}{8\pi^2} \int d^3 r \left\{ (\partial_\mu \chi_1)^2 + 8\Lambda^2 \chi_2^2 \right.$$

$$\left. - \frac{M^4}{32\Lambda^4} \sin(2\chi_1) \sin(2\chi_2) \right\}. \tag{56}$$

We now minimize with respect to χ_2 to obtain

$$\begin{aligned}
\chi_2 &= \frac{M^4}{256\Lambda^6} \sin(2\chi_1) \cos(2\chi_2) \\
&\simeq \frac{M^4}{256\Lambda^6} \sin(2\chi_1) + \mathcal{O}\left(\frac{M^{12}}{\Lambda^{12}}\right).
\end{aligned} \tag{57}$$

Substituting into the action we finally get

$$S_{sg} = 4 \frac{e^2 \epsilon_m}{8\pi^2} \int d^3 r \left\{ (\partial_\mu \chi_1)^2 - \frac{M^8}{8192\Lambda^6} \cos(4\chi_1) \right\} \tag{58}$$

To summarize, we see that in general instanton effects lead to a Sine-Gordon action of the form

$$S_{sg} = \frac{e^2 \epsilon_m f(S)}{8\pi^2} \int d^3r \left\{ (\partial_\mu \chi)^2 - \frac{M^2}{2^{\theta(f(S)-3/2)}} \left(\frac{M^2}{16\Lambda^2} \right)^{f(S)-1} \cos(f(S)\chi) \right\} \quad (59)$$

where

$$f(S) = \begin{cases} 1 & \text{for even integer } S \\ 2 & \text{for odd integer } S \\ 4 & \text{for half odd integer } S \end{cases} \quad (60)$$

Finally, if we make the shift $\chi \rightarrow \chi \sqrt{4\pi^2/[e^2 \epsilon_m f(S)]}$, we obtain

$$S_{sg} = \int d^3r \left\{ \frac{1}{2} (\partial_\mu \chi)^2 - M^2(S) \cos(\chi h(S)) \right\} \quad (61)$$

with

$$h(S) = \sqrt{\frac{4\pi^2 f(S)}{e^2 \epsilon_m}} \\ M^2(S) = \frac{2(ma)^{2\rho}}{a^3} f(S) 2^{-\theta(f(S)-3/2)} \times \left(\frac{\pi^2 (ma)^{2\rho}}{e^2 \epsilon_m a} \right)^{f(S)-1} \quad (62)$$

We see that the effects of instantons are described by a Sine-Gordon theory. The conclusion that instanton effects can be described by a Sine-Gordon theory is true for all microscopic spins. The only difference is in the mass of the Sine-Gordon theory we just wrote and in the factor inside the cosine of the Sine-Gordon theory. Hence the long distance confinement consequences of tunneling effects are equivalent independent of the microscopic spins with just irrelevant numerical differences. Even though we did not discuss it in detail here, the Berry phases do make a difference for the paramagnetic ground state, which leads to breaking of lattice symmetries^{14,18}. They make the ground state quadruply degenerate for half-odd integer spins, doubly degenerate for odd integer spins, and a non-degenerate valence bond solid for even integer spins. *But as far as the instantons the long distance confinement physics is the same regardless of microscopic spins.*

B. Confinement of z Spinons in the Quantum Paramagnetic Phase

In order to see that the effects of instanton fluctuations lead to confinement, we now closely follow Polyakov and calculate the correlation or Green's function between

electromagnetic fields. The electromagnetic field is defined as before

$$H_\mu(x) = \frac{1}{2} \epsilon_{\mu\nu\lambda} F_{\nu\lambda}(x). \quad (63)$$

The instantons, or the instanton charge density, acts as a source for these electromagnetic fields via

$$H_\mu(x) = \frac{1}{2} \int d^3y \frac{(x-y)_\mu}{|\vec{x}-\vec{y}|^3} \rho(\vec{y}) \quad (64)$$

or in momentum space

$$H_\mu(k) = 2\pi i \frac{k_\mu}{k^2} \rho(k). \quad (65)$$

It proves very convenient to introduce sources for our instantons. In particular we calculate

$$\langle e^{i \int \rho(x) \eta(x) d^3x} \rangle = \frac{Z[\eta(x)]}{Z[0]}, \quad (66)$$

where

$$\rho(x) = \sum_a q_a \delta(x - x_a) \\ Z_{inst}[\eta] = \int \mathcal{D}\chi \exp \left\{ - \int d^3r \left[\frac{1}{2} [\partial_\mu (\chi - \eta)]^2 - M^2(S) \cos(\chi h(S)) \right] \right\} \quad (67)$$

Correlation functions of ρ can be derived from the partition function by taking derivatives with respect to η : $\partial^{(n)} Z_{inst} / \partial \eta^{(n)}$. In particular, we obtain

$$\langle \rho(k) \rho(-k) \rangle = k^2 - k^4 \langle \chi(k) \chi(-k) \rangle \quad (68)$$

The correlation function for the electromagnetic fields is given by

$$\langle H_\mu(k) H_\nu(-k) \rangle = \langle H_\mu(k) H_\nu(-k) \rangle^{(0)} + \frac{k_\mu k_\nu}{k^4} \langle \rho(k) \rho(-k) \rangle \quad (69)$$

where $\langle H_\mu(k) H_\nu(-k) \rangle^{(0)}$ is the Green's function without instantons. With the appropriate screening, this Green's function is

$$\langle H_\mu H_\nu \rangle^{(0)} = \frac{1}{\epsilon_m k^2} (k^2 \delta_{\mu\nu} - k_\mu k_\nu) \\ = \frac{1}{\epsilon_m} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \quad (70)$$

The $k = 0$ pole reflects the masslessness of the photon. Now we calculate the instanton-instanton density correlation function. Before doing so we note that in the dilute gas approximation the χ coupling constant is extremely small ($ma \ll 1$) and thus provides only a small renormalization without changing the qualitative behavior of the theory. Hence the Sine-Gordon cosine potential may

be approximated to second order in χ to high accuracy and

$$\langle \chi(k) \chi(-k) \rangle \simeq \frac{1}{k^2 + (h(S)M(S))^2}. \quad (71)$$

We find that the instanton charge density correlator is given by

$$\begin{aligned} \langle \rho(k) \rho(-k) \rangle &= k^2 - \frac{k^4}{(h(S)M(S))^2 + k^2} \\ &= \frac{(h(S)M(S))^2 k^2}{k^2 + (h(S)M(S))^2}. \end{aligned} \quad (72)$$

The electromagnetic propagator becomes

$$\begin{aligned} \langle H_\mu(k) H_\nu(-k) \rangle &= \\ \frac{1}{\epsilon_m} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{k_\mu k_\nu}{k^2} \frac{(h(S)M(S))^2}{k^2 + (h(S)M(S))^2} \right] \\ &= \frac{1}{\epsilon_m} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + (h(S)M(S))^2} \right] \end{aligned} \quad (73)$$

We see from the pole in the last term that the photon acquired a mass $M_p = h(S)M(S)$ without spontaneous breaking of $U(1)$ symmetry. This is a sign of confinement and that the ground state is a gauge singlet. If the reader is unsatisfied with this correct, but indirect conclusion, we can check that the theory indeed confines by calculating that the Wilson loop^{22,23,24}

$$F[C] \equiv e^{-\mathcal{W}[C]} \equiv \langle e^{i \oint A_\mu dx_\mu} \rangle. \quad (74)$$

gives an area law^{22,23}. Explicitly, using Stokes' theorem we have

$$F[C] = \langle e^{i \oint A_\mu dx_\mu} \rangle = \langle e^{i \int_A H_\mu dS_\mu} \rangle \quad (75)$$

which, using equation (64) can be written as

$$F[C] = \langle e^{i \int \eta(x) \rho(x) d^3x} \rangle \quad (76)$$

with

$$\eta(x) = \frac{1}{2} \int_A dS_y \cdot \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3}. \quad (77)$$

Using (66) and (67) we find

$$\begin{aligned} F[C] &= \frac{1}{Z[0]} \int \mathcal{D}\chi \exp \left\{ - \int d^3r \left[\frac{1}{2} [\partial_\mu (\chi - \eta)]^2 \right. \right. \\ &\quad \left. \left. - M^2(S) \cos(\chi h(S)) \right] \right\}. \end{aligned} \quad (78)$$

In the saddle point approximation we obtain

$$\begin{aligned} F[C] &\sim \exp \left\{ - \int d^3r \left[\frac{1}{2} [\partial_\mu (\chi_{cl} - \eta)]^2 \right. \right. \\ &\quad \left. \left. - M^2(S) \cos(\chi_{cl} h(S)) \right] \right\}. \end{aligned} \quad (79)$$

The classical field is obtained by solving the equation

$$\partial^2 (\chi_{cl} - \eta) = M^2(S) h(S) \sin(\chi_{cl} h(S)) \quad (80)$$

which takes the form

$$\begin{aligned} \partial^2 \chi_{cl} &= 2\pi \delta^1(z) \theta_A(xy) + M^2(S) h(S) \sin(\chi_{cl} h(S)) \\ \theta_A(xy) &= \begin{cases} 1 & x, y \in A \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \quad (81)$$

This has solution

$$\chi_{cl}(z) = \begin{cases} 4 \arctan(e^{-M(S)h(S)z}), & z > 0 \\ -4 \arctan(e^{M(S)h(S)z}), & z < 0 \end{cases}. \quad (82)$$

It now immediately follows that

$$\begin{aligned} F[C] &\simeq e^{-\gamma(S)A} \\ \gamma(S) &= \frac{1}{2} \int_{-\infty}^{\infty} dz (\chi_{cl} - \eta)(z) (\chi_{cl}'' - \eta'')(z) \\ &\quad + \int_{-\infty}^{\infty} dz M^2(S) \cos(\chi_{cl}(z) h(S)) \end{aligned} \quad (83)$$

where A is the area of the xy -plane.

We see that the Wilson loop satisfies the area law. The interaction energy between charges is given by

$$E(R) = \gamma(S)R \quad (84)$$

and the theory confines for all values of the microscopic spins. The force between spinons is constant and equal to the “string tension” $\gamma(S)$. The scale below which the theory confines $1/\xi(S) = M(S)h(S)$ is spin dependent but nonzero. Basically spinons are confined at length scales larger than $\xi(S)$ and are not relevant to the physics of the system. The low energy excitations are spin 1 triplons ($z^\dagger \vec{\sigma} z$) and a spin 0 collective mode corresponding to the Sine-Gordon field χ .

V. QUANTUM CRITICAL DECONFINEMENT OF SPINONS

We saw in the previous section that spinons are confined and that the confinement arose from the topological tunneling events originating from the compactness of the theory. On the other hand, it has recently been suggested that there is a new class of quantum critical points whose properties are controlled by the deconfinement of spinons rather than fluctuations of the order parameter^{5,6}. In particular, it was proposed that this new kind of quantum critical points occur in 2 + 1 D antiferromagnets. Finally, it was further proposed that, because of the dependence on microscopic spins of the Berry phases, the deconfinement occurs only for spin 1/2 systems⁶. In the present section, we will study how deconfinement occurs and under which conditions.

The quantum critical point occurs when the spinons become massless, $m = 0$. As we saw above, the only dependence on the microscopic spins, as far as long distance confinement properties, appears in the parameter of the Sine-Gordon theory that describes the nontrivial

tunneling effects in 2 + 1 D antiferromagnets. As shown in the previous section, when $m > 0$ antiferromagnets confine spinons for all microscopic spins. The presence of confinement is characterized by the nonzero value of the photon mass or confinement scale

$$M_p(S) = M(S)h(S) = \sqrt{\frac{8\pi^2 (ma)^{2\rho}}{e^2 \epsilon_m a^3} f^2(S) 2^{-\theta(f(S)-3/2)} \left(\frac{\pi^2 (ma)^{2\rho}}{4e^2 \epsilon_m a} \right)^{f(S)-1}}, \quad (85)$$

such that confinement occurs for all energies *smaller* than $M_p(S)$. It now follows that the confinement energy scale vanishes at the quantum critical point for all values of the microscopic spin. *Therefore, spinons are deconfined at the quantum critical point independent of the microscopic spin.* On the other hand, there is a dependence of microscopic spins on the approach to the critical point. For half-odd integer spin systems the confinement length scale $\xi(S) = 1/M_p(S)$ diverges faster than for odd integer spin systems. The confinement length scale diverges faster for odd integer spin systems than for even integer spin systems. Therefore the only dependence on the microscopic spin is how fast one reaches the deconfined quantum critical point, but deconfinement occurs for all microscopic spins.

The conclusion that deconfinement is independent of microscopic spins is new and unexpected. Therefore we will check that this is indeed so with more care. At face value the only conclusion that can be reached is that at $m = 0$, $M(S) = 0$. Let's calculate the Wilson loop for the case of $M(S) \rightarrow 0$ and see if the theory confines or not. As we saw in the previous section, to calculate the Wilson loop we need to evaluate

$$F[C] \sim \exp \left\{ - \int d^3 r \left[\frac{1}{2} [\partial_\mu (\chi_{cl} - \eta)]^2 - M^2(S) \cos(\chi_{cl} h(S)) \right] \right\}. \quad (86)$$

where

$$\partial^2(\chi_{cl} - \eta) = M^2(S)h(S) \sin(\chi_{cl} h(S)). \quad (87)$$

When $m = 0$, $M(S) = 0$ and we have

$$F[C] \sim \exp \left\{ - \frac{1}{2} \int d^3 r [\partial_\mu (\chi_{cl} - \eta)]^2 \right\} \quad (88)$$

with

$$\partial^2(\chi_{cl} - \eta) = 0. \quad (89)$$

Now the Wilson loop is calculated very straightforwardly by integrating by parts to obtain

$$F[C] \sim \exp \left\{ \frac{1}{2} \int d^3 r (\chi_{cl} - \eta) \partial^2 (\chi_{cl} - \eta) \right\} = 1. \quad (90)$$

Since $F[C] = e^{-ER}$, the long distance force between spinons is zero and they are deconfined. This result is independent of microscopic spins as $M(S) \rightarrow 0$ for all microscopic spins.

There is one last but equivalent way to see that deconfinement occurs independent of microscopic spin, i.e. that the instantons or compactness effects disappear at the quantum critical point. If we go back to the partition function from which all the confinement physics followed

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}z \mathcal{D}z^\dagger \mathcal{D}A_\mu e^{-S_{zA}} \mathcal{Z}_{inst} \\ S_{zA} &= \frac{1}{4e^2 \epsilon_m} \int d^3 r F_{\mu\nu}^2 \\ \mathcal{Z}_{inst} &= \sum_{K,q_s} \frac{\zeta^K}{K!} \prod_{s=1}^K \left[\sum_{R_a} \int_0^\beta \frac{d\tau_s}{a} \right] e^{-S_{inst}} \\ S_{inst} &= \frac{\pi}{2e^2 \epsilon_m} \sum_{s \neq t} \frac{q_s q_t}{[(\vec{R}_s - \vec{R}_t)^2 + (\tilde{\tau}_s - \tilde{\tau}_t)^2]^{1/2}} \\ &\quad + i \sum_s \pi S \zeta_s q_s \\ \zeta &= e^{-S_{cl}} = \left[\frac{m}{\Lambda} \right]^{2\rho(q)}. \end{aligned} \quad (91)$$

ζ is the instanton fugacity as calculated before²¹ and $\rho(q) > 0$ except when we have no instantons, where by definition $\rho(0) \equiv 0$. $\zeta = 1$ for this last case. It now follows immediately that at the critical point $\zeta = 0$ except in the absence of instantons, $K = 0, q = 0$. Therefore, $\mathcal{Z}_{inst} = 1$, there are no Berry phase terms and no monopole events and the theory is deconfined independent of microscopic spins.

VI. SPINON DECONFINED CRITICAL THEORY

As we obtained in the previous section, at the quantum critical point, when the spinon mass m vanishes, the instanton and Berry phase terms disappear from the theory and the quantum critical point corresponds to massless spinons coupled to an emergent $U(1)$ gauge field with no

effects of topology. The emergent gauge group is noncompact at criticality. We had calculated before the effects of the spinons on the emergent photon and found that when the spinons were integrated out, we obtain “dielectric screening” of the emergent electromagnetic fields. This is easily understood as the spinons have a gap and hence the theory is a “semiconductor” as far as the emergent electromagnetic properties. The spinons are “semimetallic” at criticality and we need to calculate the screening properties in this case anew.

The critical theory is described by the action

$$S = \frac{2}{ga} \int d^3r |(\partial_\mu - iA_\mu)z|^2 + \int d^3r \left\{ i\delta\lambda(|z|^2 - 1) + \frac{1}{4e^2} F_{\mu\nu}^2 \right\}. \quad (92)$$

As before, $\delta\lambda$ is a Lagrange multiplier to enforce the constraint $|z|^2 = 1$. Just as the gauge fields A_μ acquire dynamics, spinon fluctuations give dynamics to the Lagrange multiplier $\delta\lambda$. The dynamics for $\delta\lambda$ follows from the same one loop spinon fluctuations that gave dynamics to the gauge fields. To one loop order

$$\langle \delta\lambda(k) \delta\lambda(-k) \rangle \simeq \frac{1}{\pi^2} \left(\frac{g}{2\Lambda} \right)^2 \frac{1}{k}. \quad (93)$$

This term is the same as would be obtained from expanding the exponential of the action

$$\frac{1}{\pi^2} \left(\frac{g}{2\Lambda} \right)^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k} \delta\lambda(k) \delta\lambda(-k). \quad (94)$$

The $1/k$ means that $\delta\lambda$ fluctuations will be suppressed at long wavelengths and hence irrelevant to the low energy physics. For completeness, we keep these terms in the effective action, which leads to a $\delta\lambda$ propagator

$$\pi^2 \left(\frac{2\Lambda}{g} \right)^2 k. \quad (95)$$

Now we must calculate the screening effect of the massless spinons. This is easily done via the renormalization group by integrating high energy spinon degrees of freedom and computing their effects on the renormalization of the electric charge. In this way we obtain the beta function

$$\tilde{\beta}_{e^2} = \mu \frac{\partial}{\partial \mu} e_\mu^2 = \frac{e_\mu^4}{\pi^2 \mu}. \quad (96)$$

This function is not dimensionless as usual because in $2 + 1$ dimensions the electric charge has dimension of square root of mass or momentum. This equation can be integrated easily and yields the renormalized charge at scale $\mu \ll \Lambda$

$$e_\mu^2 \simeq \pi^2 \mu. \quad (97)$$

Therefore, from $e_\mu^2 = e^2/\epsilon_m$, we obtain the screening factor at scale $\mu = k$ in momentum space

$$\epsilon_m = \frac{e^2}{\pi^2 k}. \quad (98)$$

Spinons interact through photon exchange. The effective interaction between spinons is given by the photon-photon Green’s function, which we calculate in the Lorentz gauge

$$V_{\mu\nu} = \langle A_\mu(k) A_\nu(-k) \rangle = \frac{\pi^2}{e^2 k} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right]. \quad (99)$$

which in real space is formally written as

$$V_{\mu\nu}(x - x') = \frac{\pi^2}{e^2 (x - x')^2} \left[\delta_{\mu\nu} - \partial_\mu \frac{1}{\partial^2} \partial_\nu \right]. \quad (100)$$

Therefore, the effective theory at criticality is described by the action

$$S = \int d^3r \left[\frac{2\Lambda}{g} |\partial_\mu z|^2 + i\delta\lambda(|z|^2 - 1) \right] - \frac{1}{2} \left(\frac{2\Lambda e}{g} \right)^2 \int d^3r d^3r' J_\mu(r) V_{\mu\nu}(r - r') J_\nu(r') + \int d^3r J^a(r) z^\dagger(r) \sigma^a z(r) \quad (101)$$

where

$$J_\mu(r) = i(z^\dagger \partial_\mu z - z \partial_\mu z^\dagger). \quad (102)$$

We have added an external source $J^a(r)$ that couples to the Néel field $n^a = z^\dagger \sigma^a z$ because we want to study the $\langle n^a(r) n^b(r') \rangle$ correlator.

In order to analyze the critical action, we introduce a Hubbard-Stratonovich real vector field B_μ to write an action linear rather than quadratic in J_μ . The action for the nonlinear sigma model at criticality is then

$$S = \int d^3r \left[\frac{2\Lambda}{g} |\partial_\mu z|^2 + i\delta\lambda(|z|^2 - 1) \right] + \frac{1}{2} \int d^3r d^3r' B_\mu(r) \tilde{V}_{\mu\nu}^{-1}(r - r') B_\nu(r') + \frac{2\Lambda e}{g} \int d^3r B_\mu(r) J_\mu(r) + \int d^3r J^a(r) z^\dagger(r) \sigma^a z(r). \quad (103)$$

with

$$\tilde{V}_{\mu\nu}^{-1}(k) = \frac{e^2 k}{\pi^2} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \quad (104)$$

Of course, in the partition function we must integrate over $z, z^\dagger, \delta\lambda$ and B_μ . The B_μ ’s are gauge fields with the appropriate gauge invariance as can be seen from their action. We can now integrate the spinon fields z to obtain the partition function

$$Z = \int \mathcal{D}B_\mu \mathcal{D}\delta\lambda e^{-S_{\text{eff}}} \quad (105)$$

where S_{eff} is, in momentum space for convenience

$$S_{\text{eff}} = \exp \left(- \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} B_\mu(-k) \tilde{V}_{\mu\nu}^{-1}(k) B_\nu(k) - i\delta\lambda \right] - \text{Tr} \ln M \right) \quad (106)$$

The operator M is

$$\begin{aligned} M(q_1, q_2) &= \left(i \frac{g}{2\Lambda} \delta\lambda + q_1^2 \right) \delta(q_1 - q_2) \\ &+ B_\mu(-q_1 + q_2) \left[\frac{q_1 + q_2}{2} \right]_\nu \\ &+ B_\mu(q_1 - q_2) \left[\frac{q_1 + q_2}{2} \right]_\nu + J^a(q_1 - q_2) \sigma^a. \end{aligned} \quad (107)$$

We can now move to calculate the Néel field correlator

$$\begin{aligned} \langle n^a(k) n^b(-k) \rangle \delta(k + q) &\equiv \langle n^a(k) n^b(q) \rangle \\ &= \frac{1}{Z} \frac{\partial^2 Z}{\partial J^a(k) \partial J^b(q)} \\ &= \frac{1}{Z} \text{Tr}(\sigma^a \sigma^b) \int \mathcal{D}B_\mu \mathcal{D}\delta\lambda \int \frac{d^3 q_1 d^3 p}{(2\pi)^6} \\ &\times M^{-1}(q_1, p) M^{-1}(p - k, q + q_1) \Big|_{J=0} e^{-S_{\text{eff}}} \\ &= \frac{2\delta^{ab}}{Z} \int \mathcal{D}B_\mu \mathcal{D}\delta\lambda \int \frac{d^3 q_1 d^3 p}{(2\pi)^6} \\ &\times M^{-1}(q_1, p) M^{-1}(p - k, q + q_1) \Big|_{J=0} e^{-S_{\text{eff}}}. \end{aligned} \quad (108)$$

To lowest order we find

$$\begin{aligned} \langle n^a(k) n^b(-k) \rangle &= 2\delta^{ab} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2(p - k)^2} \\ &= \frac{\delta^{ab}}{k} \left(\frac{1}{8} + \frac{1}{\pi^2} \right) \simeq \frac{\delta^{ab}}{k} 0.25. \end{aligned} \quad (109)$$

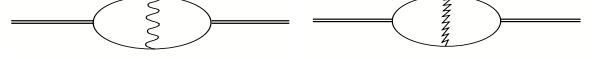
We point out that this is what is obtained from non-interacting, free spinons. That is, *the Néel field decays into two free spinons, which are then reabsorbed to reconstitute the Néel field*. We see then that *the Néel field Green's function has anomalous exponent $\eta = 1$. This is the unique consequence of decay of the Néel magnetization into free spinons at criticality*. In our model, the spinons are expected to be free as they can interact through emergent photons, but at the critical point the electromagnetic potentials acting between the charge currents generated by the spinons are screened from the Coulomb form $1/k^2$ to $1/k$, leading to a retarded interaction $\sim 1/(\vec{R}^2 + \tau^2)$, and to a Coulomb law $\sim 1/|\vec{R}|$. Hence the spinons are expected to be free. Let's see if this is indeed so. The one loop correction coming from the effective spinon interaction is given by

$$\begin{aligned} \langle n^a(k) n^b(-k) \rangle \Big|_{1 \text{ loop}} &= 8\pi^2 \delta^{ab} \int \frac{d^3 q_2 d^3 p}{(2\pi)^6} \frac{p \cdot (k + p)}{q_2(p - q_2)^2 p^2(p + k)^2(k + p - q_2)^2} \\ &\simeq \frac{\delta^{ab}}{k} 0.104329. \end{aligned} \quad (110)$$

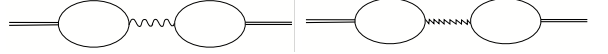
Therefore, including the interactions to first order we find that the spinons still behave as free as the anomalous exponent $\eta = 1$. The only effect of the interactions was to

renormalize the non-universal constant of proportionality multiplying $1/k$.

Our last result is a one photon interaction result, but a simple exercise in Feynman diagrams using the Feynman rules that follow from the effective action (103) shows that higher order corrections lead to corrections proportional also to $1/k$. Inclusion of Lagrange multiplier $\delta\lambda$ lines or photon lines internal to the spinon bubble



gives contributions proportional to $1/k$ by direct computation or by simple power counting. In these diagrams, the double solid lines correspond to Néel field n lines, the single solid lines corresponds to spinon z or z^\dagger lines, the wiggly line corresponds to a photon propagator and the lightning bolt corresponds to $\delta\lambda$ lines or propagators. Similarly, inclusion of further internal lines to the bubble gives contributions proportional to $1/k$. There are also corrections given by the diagrams



which give contributions proportional to $1/k$. Inclusion of higher order bubbles or corrections internal to the spinon bubble also yield contributions proportional to $1/k$. One nontrivial point which could invalidate the conclusion that $\eta = 1$ is if internal self energy corrections to the spinon propagator include $\ln k$ terms. Explicit calculation of the spinon self energy corrections shows that the log terms that are generated are multiplied by factors of k^2 which make them vanish as $k \rightarrow 0$ and are thus irrelevant to the universal low energy critical physics. Hence we see that *at criticality, spinons are not only deconfined but behave as noninteracting at long distances and lead to critical exponent $\eta = 1$ equivalent to that of decay of the Néel field into critical free spinons*.

The deconfined critical points studied and elucidated here seem to be different from the $2 + 1$ D Heisenberg critical points. It has been suggested before that these two different types of critical points might occur in $2 + 1$ D⁶. One particular suggestion is that interactions irrelevant to the Néel and quantum paramagnetic phases turn the Heisenberg critical point into a deconfined critical point and there seems to be indirect numerical evidence for such physics²⁵. In references²⁵, evidence for a continuous transition between a valence bond ordered paramagnet and its corresponding Néel ordered phase was presented in both Heisenberg and XY systems. Evidence for a relatively large value of η was also presented. It has been suggested that these transitions correspond to the deconfined critical points⁶ studied in the present work.

We have shown that if a deconfined critical point exists in $2 + 1$ D antiferromagnets, it will occur irrespective of the microscopic spin of the system, and that the confinement length will diverge faster upon approach to criticality for half-odd integer spins, next fastest for odd integer

spins, and slowest for even integer spins as a consequence of the Berry phase terms relevant to the quantum paramagnetic phase. Deconfinement occurs because instanton or monopole events vanish at criticality²¹ and together with them, the Berry phase terms vanish too¹⁴. We also find that the emergent photon at criticality is screened strongly at long distances making the spinons behave as if they were strictly free at long wavelengths. Finally, the Néel critical correlator behavior follows from decay into free spinons, which universally leads to the critical exponent $\eta = 1$. This is a diagnostic of deconfined criticality, that is, *for a free deconfined spinon critical point we predict a critical exponent η exactly equal to one.*

VII. SOME EXPERIMENTAL CONSEQUENCES OF CRITICAL SPINON DECONFINEMENT

Now that we have found the effective critical theory of deconfined critical points, we will briefly obtain some experimental consequences of the free spinon critical theory. As we have seen, the Néel critical propagator or susceptibility is given by

$$\begin{aligned} \langle n_a(\omega, \vec{k}) n_b(-\omega, -\vec{k}) \rangle &= G_{ab}(\omega, \vec{k}) \\ &\equiv \delta_{ab} G(\omega, \vec{k}) = \delta_{ab} \frac{C}{\Lambda} \frac{1}{\sqrt{\vec{k}^2 - \omega^2}} \end{aligned} \quad (111)$$

where C is a dimensionless constant. Experimental measurable quantities are usually proportional to the density of states, which at finite temperatures is given by

$$D(\omega, \vec{k}, T) = \frac{1}{\pi} \text{Im} G(\omega, \vec{k}) \frac{1}{1 - e^{-\omega/T}}. \quad (112)$$

The imaginary part is easily calculated to be

$$\text{Im} G(\omega, \vec{k}) = \frac{\theta(|\omega| - |\vec{k}|)}{\sqrt{\omega^2 - \vec{k}^2}} \quad (113)$$

to give a density of states

$$D(\omega, \vec{k}, T) = \frac{1}{\pi} \theta(|\omega| - |\vec{k}|) \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \frac{1}{(1 - e^{-\omega/T})}.$$

The dynamic structure factor is directly proportional to the density of states, $S(\omega, \vec{k}) \propto D(\omega, \vec{k}, T)$. Hence the inelastic neutron scattering intensity is given by

$$\mathcal{I}(\omega, \vec{k}) \propto \frac{1}{\pi} \theta(|\omega| - |\vec{k}|) \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \frac{1}{(1 - e^{-\omega/T})}.$$

For neutron energy losses which are a lot larger than the temperatures of the system, the dynamic structure factor is independent of T , with small corrections which are suppressed by exponentials $e^{-\omega/T}$. For neutron energy losses which are small compared to the temperatures, the dynamic structure factor is

$$\mathcal{I}(\omega, \vec{k}) \propto \frac{1}{\pi} \theta(|\omega| - |\vec{k}|) \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \frac{T}{\omega}. \quad (114)$$

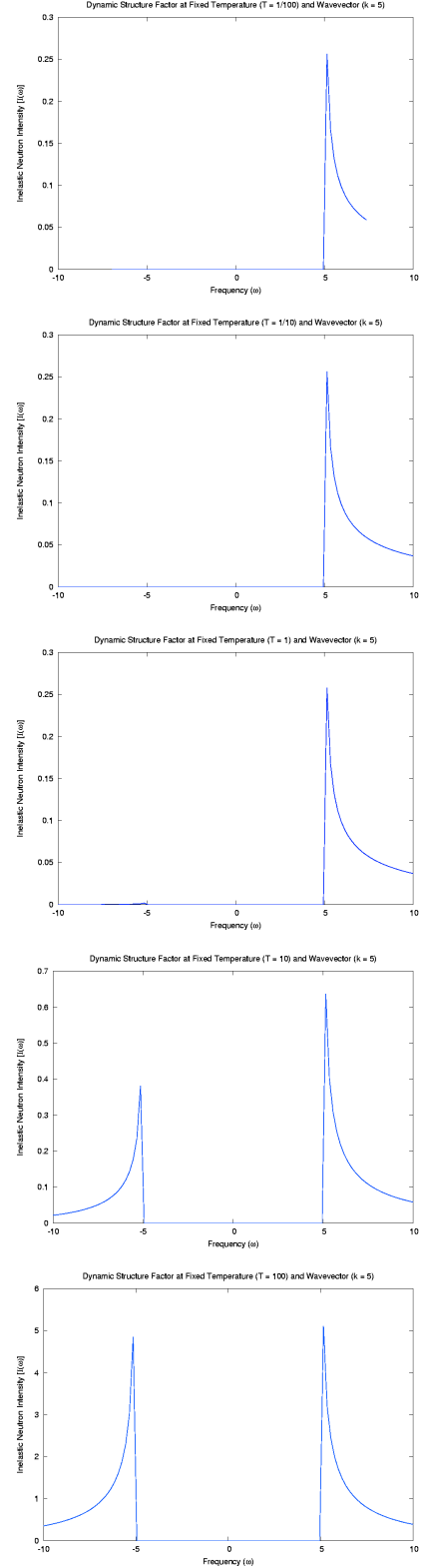


FIG. 4: Inelastic neutron scattering intensity as a function of frequency for fixed momentum and temperatures 1/100, 1/10, 1, 10 and 100 respectively.

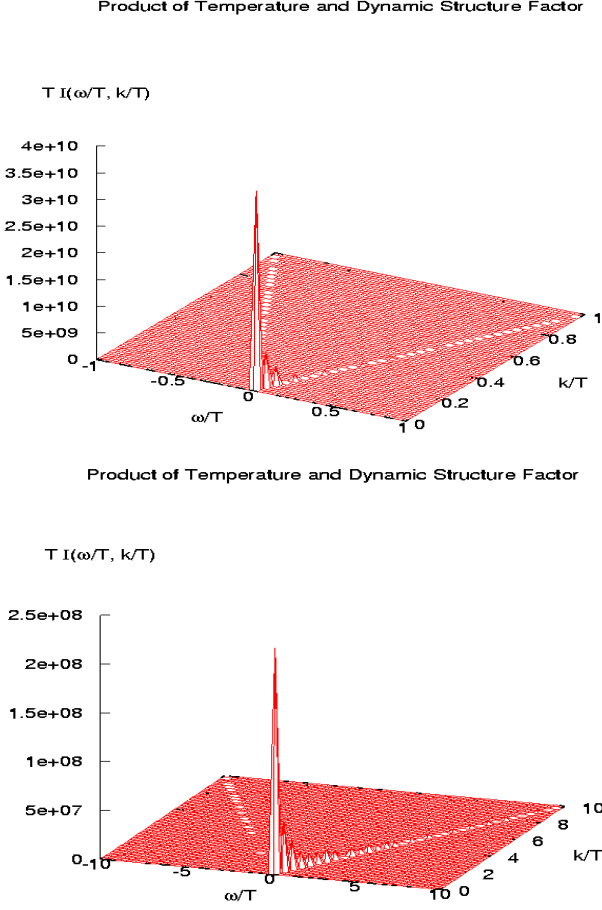


FIG. 5: Scaling of the product of the inelastic neutron scattering intensity and temperature with the ratios of frequency and momentum to temperature (ω/T and k/T).

We plot the neutron intensities as a function of ω at fixed T and $|\vec{k}|$ in figure 4. The plots can be summarized because the product of temperature and inelastic neutron scattering intensity, or structure factor, is a universal scaling function of frequency over temperature and wavevector over temperature (i.e., it satisfies a law of corresponding states), which we plot in figure 5.

Another important measurable quantity is the result of static neutron scattering experiments. The neutron intensity in this case will be proportional to the static structure factor. That is, it is proportional to the inelastic neutron response integrated over all frequencies

$$\begin{aligned} \mathcal{I}(\vec{k}) &\propto \frac{1}{\pi} \int_{-\Lambda}^{\Lambda} d\omega \theta(|\omega| - |\vec{k}|) \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \frac{1}{(1 - e^{-\omega/T})} \\ &= \frac{1}{\pi} \int_{|\vec{k}|}^{\Lambda} d\omega \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \left[\frac{1}{(1 - e^{-\omega/T})} + \frac{1}{(1 - e^{\omega/T})} \right] \\ &= \frac{1}{\pi} \int_{|\vec{k}|}^{\Lambda} d\omega \frac{1}{\sqrt{\omega^2 - \vec{k}^2}}. \end{aligned} \quad (115)$$

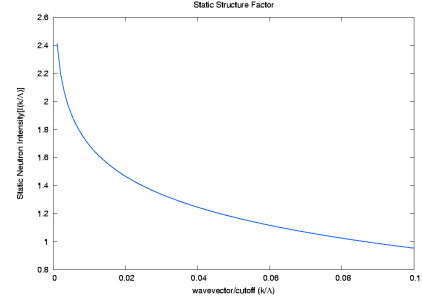


FIG. 6: Equal time neutron scattering intensity as a function of the ratio of cutoff to momentum.

The frequency integrals have been cutoff to regularize ultraviolet divergences. The static neutron intensity is temperature independent and given by

$$\mathcal{I}(\vec{k}) \propto \frac{1}{\pi} \ln \left[\frac{\Lambda}{|\vec{k}|} + \sqrt{\frac{\Lambda^2}{|\vec{k}|^2} - 1} \right] \simeq \frac{1}{\pi} \ln \left(\frac{2\Lambda}{|\vec{k}|} \right). \quad (116)$$

We plot the static neutron intensity as a function of the ratio of cutoff to wavevector transfer ($\Lambda/|\vec{k}|$) in figure 6.

In some of these antiferromagnetic systems, some of the nuclei making up the material have nonzero nuclear spin and nonzero hyperfine coupling to the electronic spins that make up the antiferromagnet. In these systems one can perform nuclear magnetic resonance (NMR) experiments. The NMR relaxation rate $1/T_1$ will be proportional to the local density of states. Therefore we have

$$\begin{aligned} \frac{1}{T_1} &\propto \frac{1}{\pi} \int d^2\vec{k} \theta(|\omega| - |\vec{k}|) \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \frac{1}{(1 - e^{-\omega/T})} \\ &= \int_0^{|\omega|} d|\vec{k}|^2 \frac{1}{\sqrt{\omega^2 - \vec{k}^2}} \frac{1}{(1 - e^{-\omega/T})} \\ &= \frac{2|\omega|}{1 - e^{-\omega/T}}. \end{aligned} \quad (117)$$

At frequencies small compared with the temperature, the NMR relaxation rate becomes independent of ω and linear in temperature. In fact for $|\omega| \ll T$

$$\frac{1}{T_1} \propto 2T \operatorname{sgn}(\omega). \quad (118)$$

For frequencies a lot larger than the temperature, the NMR relaxation rate is proportional to the magnitude of the frequency, and independent of temperature. In fact, for $|\omega| \gg T$

$$\frac{1}{T_1} \propto 2|\omega|. \quad (119)$$

We point out that the NMR relaxation rate, or local susceptibility, divided by the temperature is a universal scaling function of ω/T . We plot the NMR relaxation rate as

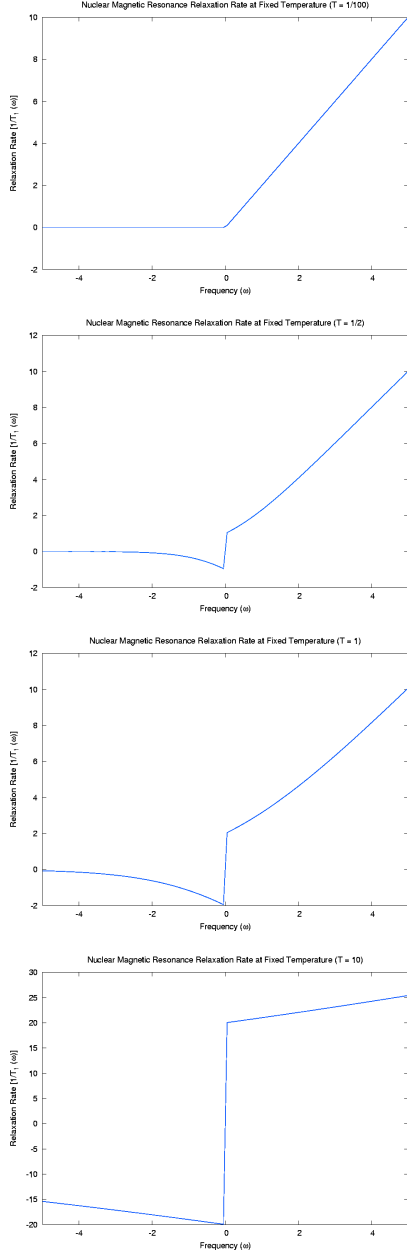


FIG. 7: Nuclear Magnetic Resonance (NMR) Relaxation Rate as a function of frequency for temperatures 1/100, 1/2, 1, and 10 respectively.

a function of frequency for fixed temperature in figure 7. We summarize the relaxation rate plots by graphing the scaling plot of the ratio of the relaxation rates to temperature, which is a universal function of ω/T , in figure 8.

We have just obtained some experimentally measurable response functions that follow from having an anomalous dimension $\eta = 1$ for the Néel magnetization propagator. Since the CP^1 mapping of the Néel magnetization into spinons proves quite cumbersome to study

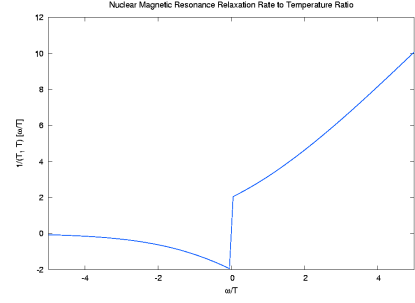


FIG. 8: Scaling of ratio of nuclear magnetic resonance (NMR) relaxation rate to temperature with ω/T .

the Néel ordered phase, it appears that using our methods we cannot say much about the critical exponents that follow as the Néel phase perishes. If the critical point where Néel order disappears is indeed a deconfined critical point as the ones studied in the present work, we can predict relations between the magnetization exponent β and the correlation length exponent ν .

The Josephson correlation length in the Néel ordered phase satisfies the renormalization group (RG) equation^{11,12,26,27,28}

$$\left[\mu \frac{\partial}{\partial \mu} + \tilde{\beta}(g) \frac{\partial}{\partial g} \right] \xi(\mu, g) = 0 \quad (120)$$

with solution

$$\xi(\mu, g) = \frac{1}{\mu} \exp \left[- \int_g^{g_c} \frac{1}{\tilde{\beta}(g')} dg' \right] \quad (121)$$

where

$$\tilde{\beta}(g) \equiv \mu \frac{\partial g}{\partial \mu} \quad (122)$$

is the usual RG beta function and g_c is the critical value of the coupling constant that separates the Néel ordered and paramagnetic phases: $\tilde{\beta}(g_c) = 0$ and $g_c > 0$. Near the critical point but on the Néel ordered phase, the correlation length scales as

$$\xi(\mu, g) \sim (g_c - g)^{1/\tilde{\beta}'(g_c)} \equiv \left[\frac{1}{(g_c - g)} \right]^\nu \quad (123)$$

with $\tilde{\beta}'(g_c) = d\tilde{\beta}/dg|_{g=g_c}$. Therefore the correlation length exponent is

$$\nu = -\frac{1}{\tilde{\beta}'(g_c)}. \quad (124)$$

Similarly, the Néel magnetization σ satisfies the RG equation^{11,12,28}

$$\left[\tilde{\beta}(g) \frac{\partial}{\partial g} + \frac{1}{2} (1 + \gamma(g)) \right] \sigma(g) = 0 \quad (125)$$

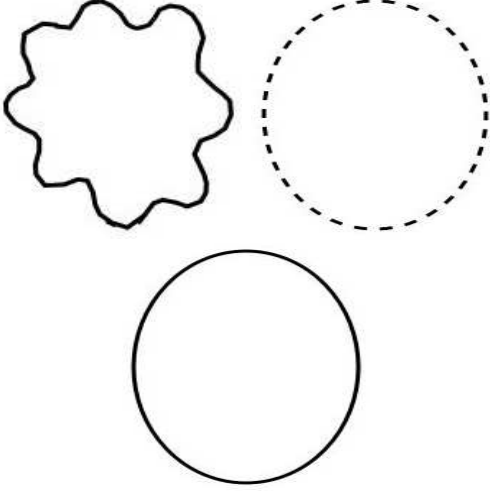


FIG. 9: Feynman diagrams representing processes contributing to the noninteracting partition function. The diagram on the top-left represents the emergent gauge field fluctuations. Lagrange multiplier $\delta\lambda$ fluctuations are represented by the diagram on the top-right, while spinon fluctuations are represented by the diagram on the bottom. Wiggly lines represent gauge fields, dashed lines represent $\delta\lambda$ fields, and straight lines represent spinons.

with solution

$$\sigma(g) = M \exp \left[-\frac{1}{2} \int_{g_c}^g \frac{(1 + \gamma(g'))}{\tilde{\beta}(g')} dg' \right]. \quad (126)$$

M is an arbitrary nonuniversal constant and $\gamma(g)$ is the anomalous dimension obtained from the magnetization renormalization factor Z via

$$\gamma(g) \equiv \mu \frac{\partial \ln Z}{\partial \mu}. \quad (127)$$

The critical anomalous dimension is then given by $\eta = \gamma(g_c)$, which is 1 for deconfined spinons. Near the critical point but on the Néel ordered phase the magnetization scales as

$$\begin{aligned} \sigma(\mu, g) &\sim (g_c - g)^{-(1+\gamma(g_c))/[2\tilde{\beta}'(g_c)]} \\ &= (g_c - g)^{-(1+\eta)/[2\tilde{\beta}'(g_c)]} \equiv (g_c - g)^\beta. \end{aligned} \quad (128)$$

Therefore the magnetization exponent is

$$\beta = -\frac{(1 + \eta)}{2\tilde{\beta}'(g_c)}, \quad (129)$$

which leads to the exponent relation

$$\beta = \frac{(1 + \eta) \nu}{2}. \quad (130)$$

For deconfined quantum critical points the correlation length exponent ν and the magnetization exponent β satisfy the relation

$$\beta = \nu \quad (131)$$

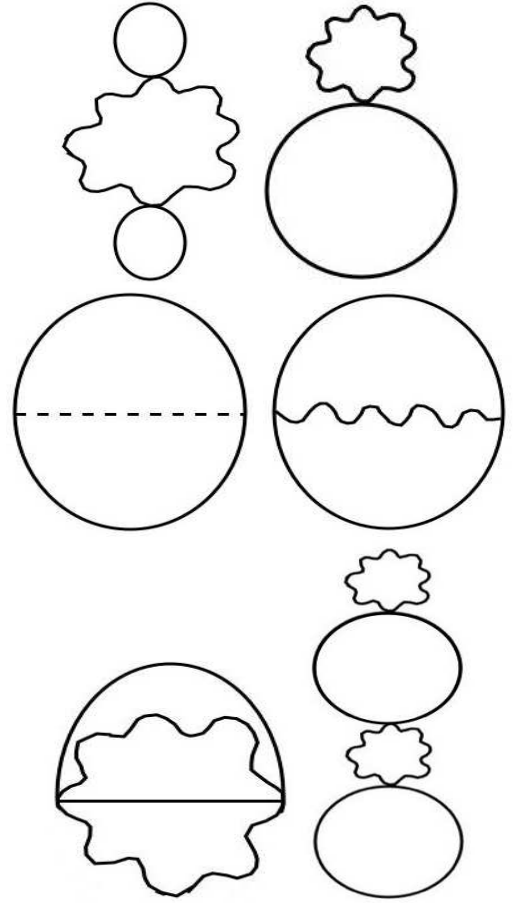


FIG. 10: Feynman diagrams representing processes contributing to the interacting partition function. These are interactions among spinons, Lagrange multiplier fields $\delta\lambda$ and emergent gauge fields. Straight lines represent spinons, wiggly lines represent emergent gauge fields, and dashed lines represent $\delta\lambda$ fields.

which is a unique prediction for deconfined critical spinons.

We can also approximate and predict the behavior of the specific heat at the quantum critical point. Since this is a finite temperature property, it is usually calculated in Euclidean time, where the imaginary time direction is finite and goes from 0 to the inverse temperature $\beta = 1/T$. The effective action (103) is separated into interacting and noninteracting parts. The noninteracting part includes all quadratic terms. The rest of the terms are grouped in the interacting part.

Before analysing the action, as with all gauge field theories, we must fix the gauge for B_μ in order to remove the gauge redundancy. This is done by the Faddeev-Popov procedure, but in this case in which we will choose the Lorentz gauge, the Faddeev-Popov determinant only provides an irrelevant renormalization constant. In this case, choosing the Lorentz gauge has the only consequence that the terms $k_\mu B_\mu$ in the action are

zero. Hence

$$\tilde{V}_{\mu,\nu}^{-1}(k) \rightarrow \frac{e^2}{\pi^2} \delta_{\mu\nu} k \quad (132)$$

In order to compute the specific heat, we must compute the free energy which being the generator of connected Green's functions, it is equal to $-T \ln Z$, where $\ln Z$ is the log of the free field partition function plus the sum of connected vacuum processes. The partition function receives contributions from the interacting and noninteracting parts of the effective action. This divides it in a product of a noninteracting Z_0 and interacting Z_I partition function, such that $Z = Z_0 Z_I$. The relevant Feynman diagrams contributing to the noninteracting partition function Z_0 are shown in Figure 9. These contributions yield a term in the specific heat which is a positive constant times T^2 as expected for noninteracting relativistic particles.

The leading contributions to the interacting partition function Z_I are shown in Figure 10. They give renormalizations of the T^2 , noninteracting specific heat. These contributions also give a low temperature correction to the specific heat coming from interactions. Such correction is proportional to a positive constant times $T^3 \ln T$. The specific heat divided by T^2 is plotted in Figure 11.

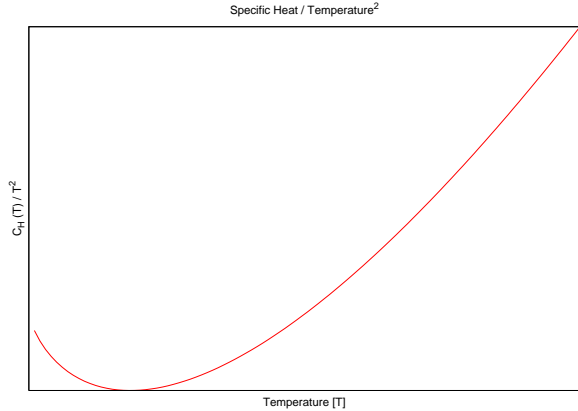


FIG. 11: Ratio of the specific heat to T^2 .

Another interesting experimental signature of deconfined, essentially free spinons at a quantum critical point is the behavior of the system in the presence of an external magnetic field. The magnetic field couples to the Néel magnetization vector and rotates it such that the sigma model partition function in the presence of the external magnetic field is

$$\begin{aligned} Z &= \int \mathcal{D}\vec{n} \delta(\vec{n}^2 - 1) e^{-S_E} \\ S_E &= iS_B + \frac{\rho_s}{2} \int_0^\beta d\tau \int d^2\vec{x} \\ &\times \left[(\partial_{\vec{x}} \vec{n})^2 + \frac{1}{c^2} \left(\partial_\tau \vec{n} - ig_B \vec{B} \times \vec{n} \right)^2 \right] \end{aligned} \quad (133)$$

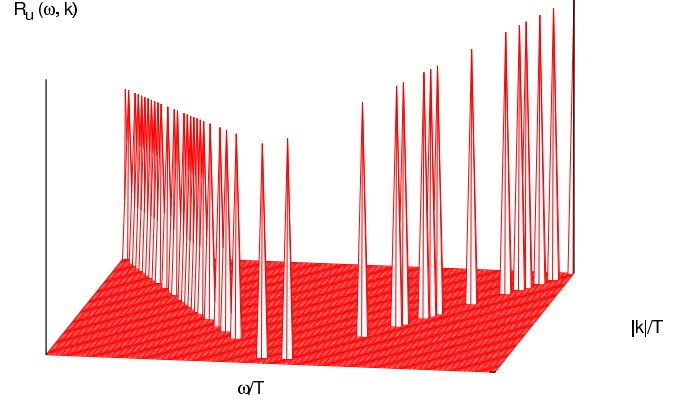


FIG. 12: Physical response of the system to an external magnetic field as a function of the ratios of frequency and momentum to temperature.

The magnetization is given by

$$\begin{aligned} M_i &= \frac{\partial \ln Z}{\partial B_i} = \frac{1}{Z} \frac{\partial Z}{\partial B_i} \Big|_{\vec{B}=0} \\ &= -\frac{1}{Z} \int \mathcal{D}\vec{n} \delta(\vec{n}^2 - 1) e^{-S_E} \frac{\partial S_E}{\partial B_i} \Big|_{\vec{B}=0} \\ &= -\left\langle \frac{\partial S_E}{\partial B_i} \right\rangle \Big|_{\vec{B}=0} \end{aligned} \quad (134)$$

and the susceptibility is

$$\begin{aligned} \chi_{ij} &= \frac{\partial M_i}{\partial B_j} = \frac{\partial^2 \ln Z}{\partial B_i \partial B_j} \Big|_{\vec{B}=0} \\ &= -\left\langle \frac{\partial S_E}{\partial B_j} \right\rangle \left\langle \frac{\partial S_E}{\partial B_i} \right\rangle \Big|_{\vec{B}=0} \\ &\quad + \left\langle \frac{\partial S_E}{\partial B_i} \frac{\partial S_E}{\partial B_j} \right\rangle \Big|_{\vec{B}=0} - \left\langle \frac{\partial^2 S_E}{\partial B_i \partial B_j} \right\rangle \Big|_{\vec{B}=0} \end{aligned} \quad (135)$$

The expectation value of the magnetization in the ground

state is zero as it consist of

$$\begin{aligned}
\frac{\partial S_E}{\partial B_i(\vec{x}, \tau)} &= \frac{\rho_s}{2c^2} \int_0^\beta d\tau' \int d^2\vec{x}' \\
&\times \frac{\partial}{\partial B_i(\vec{x}, \tau)} \left[ig_B \partial_{\tau'} \vec{n}(\vec{x}', \tau') \cdot \left(\vec{B}(\vec{x}', \tau') \times \vec{n}(\vec{x}', \tau') \right) \right. \\
&+ ig_B \left(\vec{B}(\vec{x}', \tau') \times \vec{n}(\vec{x}', \tau') \right) \cdot \partial_{\tau'} \vec{n}(\vec{x}', \tau') \\
&+ g_B^2 \left(\vec{B}(\vec{x}', \tau') \times \vec{n}(\vec{x}', \tau') \right) \cdot \left(\vec{B}(\vec{x}', \tau') \times \vec{n}(\vec{x}', \tau') \right) \left. \right] \\
&= \frac{\rho_s}{2c^2} \left[ig_B \epsilon_{ijk} \partial_\tau n_k(\vec{x}, \tau) n_j(\vec{x}, \tau) \right. \\
&+ ig_B \epsilon_{ijk} n_j(\vec{x}, \tau) \partial_\tau n_k(\vec{x}, \tau) \\
&+ g_B^2 \epsilon_{ijk} \epsilon_{klm} B_l(\vec{x}, \tau) n_j(\vec{x}, \tau) n_m(\vec{x}, \tau) \\
&+ g_B^2 \epsilon_{jkm} \epsilon_{ilm} B_j(\vec{x}, \tau) n_k(\vec{x}, \tau) n_l(\vec{x}, \tau) \left. \right] \\
&= \frac{\rho_s}{c^2} \left\{ ig_B \vec{n}(\vec{x}, \tau) \times \partial_\tau \vec{n}(\vec{x}, \tau) \right. \\
&+ g_B^2 \vec{n}(\vec{x}, \tau) \times \left[\vec{B}(\vec{x}, \tau) \times \vec{n}(\vec{x}, \tau) \right] \left. \right\}
\end{aligned} \tag{136}$$

whose expectation value at $\vec{B} = 0$ is zero for it consists of cross products of \vec{n} 's, which make them point in different directions. The correlation function of two \vec{n} 's pointing along different directions is zero.

A lengthy but straightforward calculation yields the susceptibility, which is a nonuniversal constant times ω/k

$$\chi_{ij}^u = \chi^u \delta_{ij} \sim \frac{\omega}{k^2} \delta_{ij} \tag{137}$$

The superscript u differentiates this unstaggered susceptibility from the Néel susceptibility.

The physical response to the magnetic field is proportional to the imaginary part of the retarded susceptibility. Use of the fluctuation dissipation theorem leads to

$$\begin{aligned}
R^u(\omega, \vec{k}) &\sim \frac{1}{1 - e^{-\omega/T}} \text{Im} \chi^u(\omega, \vec{k}) \\
&\sim \frac{1}{1 - e^{-\omega/T}} \omega \delta(\omega - |\vec{k}|) \\
&- \frac{1}{1 - e^{-\omega/T}} \omega \delta(\omega + |\vec{k}|) \\
&= \frac{1}{1 - e^{-\omega/T}} \frac{\omega}{T} \delta\left(\frac{\omega}{T} - \frac{|\vec{k}|}{T}\right) \\
&- \frac{1}{1 - e^{-\omega/T}} \frac{\omega}{T} \delta\left(\frac{\omega}{T} + \frac{|\vec{k}|}{T}\right).
\end{aligned} \tag{138}$$

It exhibits a temperature broadened spinon pole and is thus directly sensitive to spinon creation. We see that the response is a universal scale invariant function of ω/T and $|\vec{k}|/T$. When the energy and momentum are just right, the system takes energy from the magnetic field by shooting off spinons. This behavior is illustrated in Figure 12. At zero temperature the response is

$$R^u(\omega, \vec{k}) \sim \omega \delta(\omega - |\vec{k}|). \tag{139}$$

On the other hand, it might be hard to apply a magnetic field with precisely the right relation between momentum and energy. Thus the static response, obtained by integrating over all frequencies, might be of more relevance. It is given by

$$\begin{aligned}
R^u(\vec{k}) &\sim \frac{1}{1 - e^{-|\vec{k}|/T}} |\vec{k}| + \frac{1}{1 - e^{|\vec{k}|/T}} |\vec{k}| \\
&= T \left(\frac{1}{1 - e^{-|\vec{k}|/T}} \frac{|\vec{k}|}{T} + \frac{1}{1 - e^{|\vec{k}|/T}} \frac{|\vec{k}|}{T} \right).
\end{aligned} \tag{140}$$

and illustrated in Figure 13.

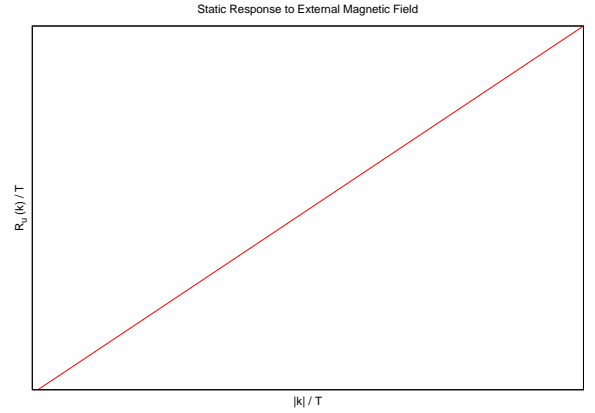


FIG. 13: Physical response of the system to an external, static magnetic field as a function of the ratio of momentum to temperature.

We see that $R^u(\vec{k})/T$ is a universal function of $|\vec{k}|/T$. At zero temperature we have

$$R^u(\vec{k}) \sim |\vec{k}|. \tag{141}$$

A similar important quantity is the response to a time dependent but uniform field, obtained by integrating over all momenta

$$R^u(\omega) \sim \omega^2 \frac{1}{1 - e^{-\omega/T}} = T^2 \left(\frac{\omega^2}{T^2} \frac{1}{1 - e^{-\omega/T}} \right). \tag{142}$$

$R^u(\omega)/T^2$ is a universal function of ω/T which is illustrated in Figure 14. At zero temperature we have

$$R^u(\omega) \sim \omega^2 \theta(\omega). \tag{143}$$

We have seen that deconfined quantum critical points are examples of new types of quantum phase transitions where the standard Wilson-Ginzburg-Landau criteria that critical properties are controlled only by order parameter fluctuation fails as has been recently suggested⁶. We have uncovered new properties and physics of such deconfined critical points as well as predicted some of their experimental features.

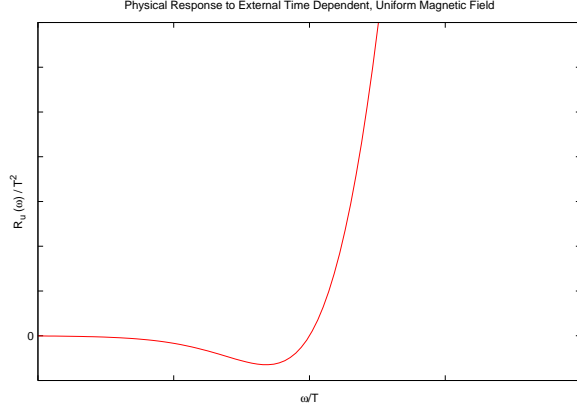


FIG. 14: Physical response of the system to an external, time dependent, uniform magnetic field as a function of the ratio of frequency and temperature.

-
- ¹ K. Wilson and J. Kogut, Phys. Rep. C **12**, 75 (1974).
 - ² J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).
 - ³ A. J. Millis, Phys. Rev. B **48**, 7183 (1993).
 - ⁴ H. V. Löhneysen, T. Pietrus, G. Portisch, H. G. Schlager, A. Schröder, M. Sieck, and T. Trappmann, Phys. Rev. Lett. **72**, 3262 (1994); A. Schröder, G. Aeppli, E. Bucher, R. Ramazashvili and P. Coleman, Phys. Rev. Lett. **80**, 5623 (1998); A. Schröder, G. Aeppli, R. Coldea, M. Adams, O. Stockert, H. V. Löhneysen, E. Bucher, R. Ramazashvili and P. Coleman, Nature **407**, 351 (2000); P. Gegenwart, J. Custers, C. Geibel, K. Neumaier, T. Tayama, K. Tenya, O. Trovarelli, and F. Steglich, Phys. Rev. Lett. **89**, 056402 (2002); J. Custers, P. Gegenwart, H. Wilhelm, K. Neumaier, Y. Tokiwa, O. Trovarelli, C. Geibel, F. Steglich, C. Pépin and P. Coleman, Nature **424**, 524 (2003).
 - ⁵ R. B. Laughlin, Adv. Phys. **47**, 943 (1998); B. A. Bernevig, D. Giuliano and R. B. Laughlin, An. of Phys. **311**, 182 (2004)
 - ⁶ T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M. P. A. Fisher, Science **303**, 1490 (2004).
 - ⁷ S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press, Cambridge, UK (1999).
 - ⁸ S. Chakravarty, B. I. Halperin and D. R. Nelson, Phys. Rev. B **39**, 2344 (1989).
 - ⁹ A. V. Chubukov, S. Sachdev and J. Ye, Phys. Rev. B **49**, 11919 (1994).
 - ¹⁰ A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).
 - ¹¹ E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. **36**, 691 (1976).
 - ¹² E. Brézin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976).
 - ¹³ F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).
 - ¹⁴ F. D. M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988).
 - ¹⁵ T. Dombre and N. Read, Phys. Rev. B **38**, 7181 (1988).
 - ¹⁶ E. Fradkin and M. Stone, Phys. Rev. B **38**, 7215 (1988); X. G. Wen and A. Zee, Phys. Rev. Lett. **61**, 1025 (1988).
 - ¹⁷ S. Sachdev, *Low Dimensional Quantum Field Theories for Condensed Matter Physicists*, Proceedings of the Trieste Summer School 1992 (World Scientific, Singapore, 1994).
 - ¹⁸ N. Read and S. Sachdev, Phys. Rev. B **42**, 4568 (1990).
 - ¹⁹ A. D'Adda, P. Di Vecchia, and M. Luscher, Nucl. Phys. B **146**, 63 (1978).
 - ²⁰ E. Witten, Nucl. Phys. B **149**, 285 (1979).
 - ²¹ G. Murthy and S. Sachdev, Nucl. Phys. B **344**, 557 (1990).
 - ²² A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
 - ²³ A. M. Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers, Chur, Switzerland (1987).
 - ²⁴ K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
 - ²⁵ A. W. Sandvik, talk presented at the 2006 APS March Meeting at Baltimore, MD and private communication (2006). Similar results, but in XY antiferromagnets were presented in A. W. Sandvik, S. Daul, R. R. P. Singh, and D. J. Scalapino, Phys. Rev. Lett. **89**, 247201 (2002).
 - ²⁶ C. G. Callan, Phys. Rev. D **2**, 1541 (1970); K. Symanzik, Comm. Math. Phys. **18**, 227 (1970).
 - ²⁷ S. Weinberg, Phys. Rev. D **8**, 3497 (1973).
 - ²⁸ J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Fourth Edition, Oxford Univ. Press, Oxford, UK (2002).